M242 – Calculus II

Unit I Exam Review – "Solutions" (Brief)

1. $\overline{PQ} = \langle 6, 3, 6\rangle$ $\quad \|\overline{PQ}\| = \sqrt{36+9+36} = \sqrt{81} = 9$

$\overline{PR} = \langle 9, -9, 0\rangle$ $\quad \|\overline{PR}\| = \sqrt{81+81} = 9\sqrt{2}$

$\overline{QR} = \langle 3, -6, -6\rangle$ $\quad \|\overline{QR}\| = \sqrt{9+36+36} = \sqrt{81} = 9$

Indeed,

$\|\overline{PQ}\|^2 + \|\overline{QR}\|^2 = 9^2 + 9^2 = 162$

$\|\overline{PR}\|^2 = (9\sqrt{2})^2 = 2\cdot 81 = 162$

So ΔPQR is a right Δ.

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2. $C(2,2,-1)$

$r = \sqrt{2^2+2^2+(-1)^2} = \sqrt{9} = 3.$

$$\boxed{(x-2)^2 + (y-2)^2 + (z+1)^2 = 9}$$

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3. $(x^2 + 2x + 1) + (y^2 - 6y + 9) + (z^2 - 4z + 4) = 22 + 1 + 9 + 4$

$$\boxed{(x+1)^2 + (y-3)^2 + (z-2)^2 = 36}$$

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4. $\vec{a} = \langle 3, 0, -4\rangle$

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5. a.) $\bar{x} + \bar{y} = \langle 1, 2, 6\rangle$

b.) $\bar{y} - \bar{x} = \langle -1, -4, 10\rangle$

c.) $3\bar{x} - 2\bar{y} = \langle 3, 9, -6\rangle - \langle 0, -2, 16\rangle = \langle 3, 11, -22\rangle$

d.) $\bar{x}\cdot\bar{y} = 1\cdot 0 + 3(-1) + (-2)(8) = 0 - 3 - 16 = -19$.

e.) $\bar{x}\times\bar{y} = \langle 24-2, 0-8, -1-0\rangle = \langle 22, -8, -1\rangle$

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6. $\|\bar{x}\| = \sqrt{(-4)^2 + 3^2 + (-1)^2} = \sqrt{26}$

$$\boxed{\bar{u} = \frac{\bar{x}}{\|\bar{x}\|} = \frac{-4}{\sqrt{26}}\vec{i} + \frac{3}{\sqrt{26}}\vec{j} - \frac{1}{\sqrt{26}}\vec{k}}$$

7. Let $\bar{x} = \langle 6, 9, -2 \rangle$, $\|\bar{x}\| = \sqrt{36+81+4} = \sqrt{121} = 11$

$$\bar{u} = \frac{\bar{x}}{\|\bar{x}\|} = \left\langle \frac{6}{11}, \frac{9}{11}, \frac{-2}{11} \right\rangle$$

$$\boxed{\bar{y} = 2\bar{u} = \left\langle \frac{12}{11}, \frac{18}{11}, \frac{-4}{11} \right\rangle}$$

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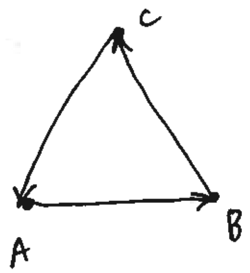
8. 

$\vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$

because the initial and terminal point coincide.

(Be sure that you can explain this well in your own words)

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9. $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta = 40 \cdot 90 \cdot \cos\left(\frac{3\pi}{4}\right) = 3600 \cdot \frac{\sqrt{2}}{2} = 1800\sqrt{2}$.

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10. $\vec{a} \cdot \vec{b} = -24 - 6 = -30$ $\qquad \theta = \cos^{-1}\left(\frac{-30}{36}\right) = \boxed{\cos^{-1}\left(\frac{-5}{6}\right)}$

$\|\vec{a}\| = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$

$\|\vec{b}\| = \sqrt{0+36+36} = 6\sqrt{2}$

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11. $\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a} = \frac{30-7-6}{36+49+36} \langle 6, 7, -6 \rangle = \frac{17}{121} \langle 6, 7, -6 \rangle = \left\langle \frac{102}{121}, \frac{119}{121}, \frac{-102}{121} \right\rangle$

12.

$\|\vec{a}+\vec{b}\|^2 = (\vec{a}+\vec{b}) \cdot (\vec{a}+\vec{b}) = \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$

$= \|\vec{a}\|^2 + 2\vec{a} \cdot \vec{b} + \|\vec{b}\|^2$

similarly, $\|\bar{a} - \bar{b}\|^2 = \quad \cdots \quad = \|\vec{a}\|^2 - 2\vec{a} \cdot \vec{b} + \|\vec{b}\|^2$

Therefore,

$\|\vec{a}+\vec{b}\|^2 + \|\vec{a}-\vec{b}\|^2 = 2\|\vec{a}\|^2 + 2\|\vec{b}\|^2$ ∎

13. $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 7 & -6 \\ 5 & -1 & 1 \end{vmatrix} = \langle 7-6, -30-6, -6-35 \rangle = \langle 1, -36, -41 \rangle$

$(\vec{a}\times\vec{b})\cdot\vec{a} = \langle 1,-36,-41\rangle \cdot \langle 6,7,-6\rangle = 6-252+246 = 252-252 = 0$ ✓

$(\vec{a}\times\vec{b})\cdot\vec{b} = \langle 1,-36,-41\rangle \cdot \langle 5,-1,1\rangle = 5+36-41 = 41-41 = 0$ ✓

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14. $\overline{AB} = \langle -4, 4, 0\rangle$

$\overline{AC} = \langle 9, 3, 0\rangle$

Area $= \| \overline{AB} \times \overline{AC} \| = \| \langle 0, 0, -12-36\rangle \| = \| \langle 0,0,-48\rangle \| = 48.$

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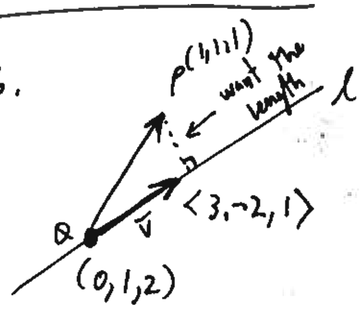
15. $\text{vol} = \left| \begin{vmatrix} 1 & 3 & 2 \\ 3 & -3 & 6 \\ -2 & 0 & 1 \end{vmatrix} \right| = \left| 1(-3) - 3(\underset{+12}{3}) + 2(-6) \right| = \left| -3-45-12 \right| = |-60| = 60.$

16.



$\boxed{\text{dist} = \| \overline{QP} - \text{proj}_{\vec{v}} \overline{QP} \|}$

$\overline{QP} = \langle 1, 0, -1\rangle \qquad \overline{QP}\cdot\vec{v} = 3-1 = 2.$

$\vec{v} = \langle 3, -2, 1\rangle \qquad \|\vec{v}\|^2 = 9+4+1 = 14$

$\text{proj}_{\vec{v}} \overline{QP} = \frac{1}{7}\vec{v} = \left\langle \frac{3}{7}, \frac{-2}{7}, \frac{1}{7} \right\rangle$

$\overline{QP} - \text{proj}_{\vec{v}} \overline{QP} = \langle 1,0,-1\rangle - \left\langle \frac{3}{7}, \frac{-2}{7}, \frac{1}{7}\right\rangle = \left\langle \frac{4}{7}, \frac{-2}{7}, \frac{-8}{7} \right\rangle$

$\| \overline{QP} - \text{proj}_{\vec{v}} \overline{QP} \| = \frac{1}{7}\sqrt{16+4+64} = \frac{\sqrt{84}}{7} = \frac{4\sqrt{21}}{7} = \frac{4\sqrt{3}}{\sqrt{7}}$

So the distance is $\text{dist} = \frac{\sqrt{84}}{7}$ or some simplification.

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17. $\vec{v} = \vec{PQ} = \langle 3, \tfrac{1}{2}, -5 \rangle$, direction vector

vector equation:

$$\vec{r}(t) = \langle 3t, \tfrac{1}{2} + \tfrac{1}{2}t, 1-5t \rangle$$

Parametric equations:
$$\begin{cases} x = 3t \\ y = \tfrac{1}{2} + \tfrac{1}{2}t \\ z = 1-5t \end{cases}$$

Symmetric Equations:
$$\frac{x}{3} = \frac{y-\tfrac{1}{2}}{\tfrac{1}{2}} = \frac{z-1}{-5}$$

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18. $\left.\begin{array}{l} \vec{N_1} = \langle 8, -4, 12 \rangle \\ \vec{N_2} = \langle 4, -2, 5 \rangle \end{array}\right\}$ not parallel

Intersecting? : $12 + 8t = 1 + 4s$

$\Rightarrow s = \dfrac{8t+11}{4}$

$16 - 4t = 3 - 2s$
$= 3 - 2\left(\dfrac{8t+11}{4}\right)$
$= 3 - 4t - \dfrac{11}{2}$
$= -4t - \dfrac{5}{2}$

$\Rightarrow 16 = -5/2$ ✗ (Nonsense).

Therefore the lines are not intersecting and are not parallel. Hence, skew.

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19. $\vec{PQ} = \langle 3, -2, -4 \rangle$
$\vec{PR} = \langle 1, -6, -5 \rangle$

$\vec{n} = \vec{PQ} \times \vec{PR} = \langle 10-24, -4+15, -18+2 \rangle = \langle -14, 11, -16 \rangle$

$\vec{r}_0 = \langle 4, 0, -1 \rangle$

$-\vec{n} \cdot \vec{r}_0 = -\langle -14, 11, -16 \rangle \cdot \langle 4, 0, -1 \rangle = -(-56+16) = 40$

so the eqn is $\boxed{-14x + 11y - 16z + 40 = 0}$

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20. $\text{dist} = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}} = \dfrac{|3(1) + 2(-3) + 6(2) - 5|}{\sqrt{9+4+36}} = \dfrac{|3-6+12-5|}{7} = \dfrac{4}{7}$