Name: Key

M243: Calculus II (Spring 2018)

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Unit II Exam: Chapters 6 and 7

WICHITA STATE UNIVERSITY

Read and follow all instructions. You may not use any notes or electronic devices. All you need is a pencil and your brain!

Part I: True/False [2 points each]

Neatly write ***T*** *if the statement is always true, and* ***F*** *otherwise.*

F **1.** The integral $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is convergent. p-test.

F **2.** $\lim\limits_{\theta\to\frac{\pi}{2}} \frac{1-\cos\theta}{\sin\theta} = \lim\limits_{\theta\to\frac{\pi}{2}} \tan\theta$ L'Hôpital does not apply.

T **3.** Simpson's Rule gives the exact answer for $\int_{-1}^{3} (3x^3 - 4x^2 + 7x - 9)\,dx$. $|E_{S_n}| \le \frac{K(b-a)^5}{180n^4}$ and $K \le f^{(4)}(x) = 0$.

F **4.** If $f(x) \le g(x)$ for all $x > 0$ and $\int_1^\infty f(x)\,dx$ is convergent, then $\int_1^\infty g(x)\,dx$ is also convergent.

$\int_1^\infty f(x)\,dx$ divergent $\Rightarrow \int_1^\infty g(x)\,dx$ divergent

F **5.** $\arctan(\tan(\theta)) = \theta$ for all $\theta \in \mathbb{R}$. only for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$!

Part II: Conceptual Problems [10 points each]

Complete all 3 problems in the space provided. Show ***enough*** *work, and write your work in a clear, organized fashion.*

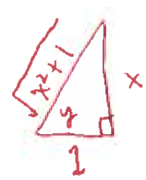
6. Derive the formula $\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{x^2+1}$.

$\frac{d}{dx}\left[y = \arctan(x)\right]$

$\Rightarrow \frac{d}{dx}\left[\tan(y) = x\right]$

$\Rightarrow \sec^2(y)\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2(y)}$



$\tan(y) = \frac{x}{1}$

$\sec(y) = \frac{H}{A} = \sqrt{x^2+1}$

$\sec^2(y) = x^2+1$

Therefore,

$\frac{dy}{dx} = \frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{x^2+1}$ $\square$

7. Write the form of the partial fraction decomposition for the rational function

$$\frac{x^2-1}{(x-3)^2(x^2+2x+4)}.$$

Do **NOT** solve for the unknown coefficients.

$$\frac{x^2-1}{(x-3)^2(x^2+2x+4)} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{Cx+D}{x^2+2x+4}$$

8. You wish to evaluate the integral

$$\int \frac{1}{\sqrt{x^2-4x-21}}\,dx$$

by using a trig substitution. Clearly indicate what substitution you make and write down the new integral. Do **NOT** evaluate the integral.

$$(x^2-4x+4)+(-21-4)$$

$$(x-2)^2-5^2$$

$$= 5^2\left(\left(\frac{x-2}{5}\right)^2-1^2\right)$$

Put $\boxed{\frac{x-2}{5} = \sec\theta}$

So, $x = 5\sec\theta + 2$

and $dx = 5\sec\theta\tan\theta\, d\theta$

The integral becomes,

$$\int \frac{5\sec\theta\tan\theta}{5\sqrt{\sec^2\theta - 1}}\,d\theta$$

$$= \boxed{\int \sec\theta\, d\theta}$$

Part III: Computational Problems [15 points each]

Complete 4 of the 5 problems in the space provided. Show <u>*enough*</u> *work. Clearly mark the one problem that you wish to* <u>*OMIT*</u>.

9. Evaluate the integral. Show *enough* work.

$$\int \frac{x^2}{x^2-4x+3}\,dx$$

long division:

$$\begin{array}{r|l} & \quad 1 \\ \hline x^2-4x+3 & x^2+0x+0 \\ & x^2-4x+3 \\ \hline & \quad 4x-3 \end{array}$$

So, $\dfrac{x^2}{x^2-4x+3} = 1 + \dfrac{4x-3}{x^2-4x+3}$

PFD:

$$\frac{4x-3}{x^2-4x+3} = \frac{A}{x-3} + \frac{B}{x-1}$$

$$4x-3 = A(x-1) + B(x-3)$$

$x=1$: $1 = -2B$

$B = \frac{-1}{2}$

$x=3$: $9 = 2A$

$A = \frac{9}{2}$

So the integral becomes:

$$\int \frac{x^2}{x^2-4x+3}\,dx = \int 1\,dx + \frac{9}{2}\int \frac{1}{x-3}\,dx - \frac{1}{2}\int \frac{1}{x-1}\,dx$$

$$= \boxed{x + \frac{9}{2}\ln|x-3| - \frac{1}{2}|x-1| + C}$$

10. Evaluate the integral. Show *enough* work.

$$\int e^{2x}\sin(x)\,dx$$

IbP:

	u	dv
$+$	e^{2x}	$\sin x\,dx$
$-$	$2e^{2x}$	$-\cos x\,dx$
$+$	$4e^{2x}$	$-\sin x\,dx$
		$\cos x\,dx$

$$\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\,dx$$

$$\Rightarrow\; 5\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2e^{2x}\sin x$$

$$\Rightarrow\; \boxed{\int e^{2x}\sin x\,dx = -\frac{1}{5}e^{2x}\cos x + \frac{2}{5}e^{2x}\sin x + C}$$

11. Determine whether the integral converges or diverges. If it converges, compute its exact value. Be sure to treat the improper integral properly.

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{x^2+1}\,dx + \lim_{u\to\infty} \int_{-u}^{0} \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \arctan(x)\Big|_0^t + \lim_{u\to\infty} \arctan(u)\Big|_{-u}^{0}$$

$$= \lim_{t\to\infty} \arctan(t) - 0 + 0 - \lim_{u\to\infty} \arctan(-u)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi.$$

12. Evaluate the integral. Show *enough* work.

$$\int_0^2 \frac{1}{\sqrt{4t^2+16}}\,dt$$

$4t^2+16 = 16\left(\left(\frac{t}{2}\right)^2+1\right)$

Put $\frac{t}{2} = \tan\theta$ $\quad \theta = \arctan\left(\frac{t}{2}\right) \Rightarrow \theta(2) = \arctan(1) = \pi/4$

$\theta(0) = \arctan(0) = 0$

$t = 2\tan\theta$

$dt = 2\sec^2\theta$

So, $\displaystyle\int_0^2 \frac{1}{\sqrt{4t^2+16}}\,dt = \int_0^2 \frac{1}{4\sqrt{\left(\frac{t}{2}\right)^2+1}}\,dt = \int_0^{\pi/4} \frac{\cancel{2}\sec^{\cancel{2}}\theta}{\cancel{4}_2\,\cancel{\sec}\theta}\,d\theta$

$\displaystyle = \frac{1}{2}\int_0^{\pi/4} \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta+\tan\theta|\Big|_0^{\pi/4}$

$= \frac{1}{2}\left(\ln|\sec\pi/4 + \tan\pi/4| - \ln|\sec 0+\tan 0|\right)$

$= \frac{1}{2}\left(\ln|\sqrt{2}+1| - \ln|1+0|\right)$

$= \frac{1}{2}\ln|\sqrt{2}+1|$

13. Evaluate the integral. Show *enough* work.

$$\int \sin^2\theta \cos^2\theta \, d\theta$$

$$\sin^2\theta = \frac{1}{2}\left(1 - \cos(2\theta)\right)$$

$$\cos^2\theta = \frac{1}{2}\left(1 + \cos(2\theta)\right)$$

$$\sin^2\theta\cos^2\theta = \frac{1}{4}\left(1 - \cos^2(2\theta)\right)$$

$$= \frac{1}{4} - \frac{1}{4}\left(\frac{1}{2}\left(1 + \cos(4\theta)\right)\right)$$

$$= \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos(4\theta)$$

$$= \frac{1}{8} - \frac{1}{8}\cos(4\theta)$$

So, $\int \sin^2\theta \cos^2\theta \, d\theta = \int \frac{1}{8}\, d\theta - \frac{1}{8}\int \cos(4\theta)\, d\theta$

$$= \boxed{\frac{1}{8}\theta - \frac{1}{32}\sin(4\theta) + C}$$