Name: Key

M344: Calculus III (Fall 2018)
Instructor: Justin Ryan
Final Exam

Read and follow all instructions. You may not use any electronic devices. You may use a single two-sided 8.5 by 11 inch page of your own hand-written notes. Each problem is worth 22 points.

1. *a.*) Define a transformation that carries the ellipse $\left(\frac{x}{3}\right)^2+\left(\frac{y}{2}\right)^2=1$ to the unit circle $u^2+v^2=1$.

$$T^{-1}: \begin{cases} u=\frac{x}{3} \\ v=\frac{y}{2} \end{cases} \qquad T: \begin{cases} x=3u \\ y=2v \end{cases}$$

b.) Compute the Jacobian, $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$, of the transformation you found in part *a*.

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right|=6$$

c.) Evaluate the integral $\iint_R \sqrt{4x^2+9y^2}\,dA$, where R is the ellipse $\left(\frac{x}{3}\right)^2+\left(\frac{y}{2}\right)^2=1$.

$$\iint_R \sqrt{(2x)^2+(3y)^2}\,dA=\iint_S \underbrace{\sqrt{(6u)^2+(6v)^2}}_{=6r \text{ in polar coord's.}}\cdot 6\,dA = 36\int_0^{2\pi}\int_0^1 r^2\,dr\,d\theta$$

$$=36\cdot 2\pi\cdot\frac{1}{3}=\boxed{24\pi}$$

2. *a.*) Give a parametrization of the curve C of intersection of the cylinder $x^2 + y^2 = 9$ and the plane $z - 2x - 3y = 0$ in $\mathbb{R}^3$. Clearly state the parameter domain.

$$\begin{cases} x = 3\cos\theta \\ y = 3\sin\theta \\ z = 6\cos\theta + 9\sin\theta \end{cases}, \quad 0 \leq \theta \leq 2\pi$$

b.) Give a parametrization of the surface S given by the portion of the plane $z - 2x - 3y = 0$ inside of the cylinder $x^2 + y^2 = 9$. Clearly state the parameter domain.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r(2\cos\theta + 3\sin\theta) \end{cases}, \quad 0 \leq r \leq 3, \quad 0 \leq \theta \leq 2\pi$$

c.) Let $\mathbf{F}(x, y, z) = \langle z, y, x \rangle$. Use your favorite method to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve in part *a*.

By Stokes' Thm, $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$

but $\text{curl}\,\vec{F} = \langle 0,0,0 \rangle = \vec{0}$, so

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \vec{0} \cdot d\vec{S} = 0.$$

3. Consider the vector field $\mathbf{F}(x,y,z) = \langle e^y - yze^{xy}, -xze^{xy} + xe^y + e^z, ye^z - e^{xy}\rangle$.

a.) Show that $\mathbf{F}$ is conservative.

$$\text{curl}\,\vec{F} = \langle e^z - xe^{xy} - (-xe^{xy} + e^z),\ -ye^{xy} - (-ye^{xy}),\ -ze^{xy} - xyze^{xy} + e^y - (e^y - ze^{xy} - xyze^{xy})\rangle$$

$$= \langle 0,0,0\rangle$$

Since $\text{curl}\,\vec{F} = \vec{0}$, then $\vec{F}$ is conservative.

b.) Find a potential function for $\mathbf{F}$.

$$f = \int e^y - yze^{xy}\,dx = xe^y - \frac{1}{y}yze^{xy} + C_1(y,z) = xe^y - ze^{xy} + C_1(y,z)$$

$$f = \int -xze^{xy} + xe^y + e^z\,dy = \frac{1}{x}(-xze^{xy}) + xe^y + ye^z + C_2(x,z) = xe^y + ye^z - ze^{xy} + C_2(x,z)$$

$$f = \int ye^z - e^{xy}\,dz = ye^z - ze^{xy} + C_3(x,y)$$

So the potential function is

$$f(x,y,z) = xe^y + ye^z - ze^{xy}$$

c.) Find the work done by the vector field $\mathbf{F}$ on a partical that moves from the point $P(1,0,0)$ to the point $Q(0,0,1)$.

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(Q) - f(P)$$

$$= (0\cdot e^0 + 0e^1 - 1\cdot e^0) - (1\cdot e^0 + 0\cdot e^0 - 0\cdot e^0)$$

$$= -1 - 1$$

$$= \boxed{-2}$$

4. Let f and g be functions in $\mathbb{R}^2$, both of whose second partial derivatives are continuous.

a.) Show that $\Delta(fg) = f\Delta(g) + g\Delta(f) + 2(\nabla f)\cdot(\nabla g)$.

$$\nabla(fg) = \langle \partial_x f \cdot g + f \cdot \partial_x g, \ \partial_y f \cdot g + f \cdot \partial_y g \rangle$$

$$\nabla \cdot \nabla(fg) = \frac{\partial}{\partial x}\left(\partial_x f \cdot g + f \cdot \partial_x g\right) + \frac{\partial}{\partial y}\left(\partial_y f \cdot g + f \cdot \partial_y g\right)$$

$$= \partial_x^2 f \cdot g + \partial_x f \cdot \partial_x g + \partial_x f \cdot \partial_x g + f \cdot \partial_x^2 g + \partial_y^2 f \cdot g + \partial_y f \partial_y g + \partial_y f \partial_y g + f \cdot \partial_y^2 g$$

$$= f\underbrace{\left(\partial_x^2 g + \partial_{yy}^2 g\right)}_{\Delta g} + g\underbrace{\left(\partial_x^2 f + \partial_y^2 f\right)}_{\Delta f} + 2\underbrace{\left(\partial_x f \cdot \partial_x g + \partial_y f \cdot \partial_y g\right)}_{\nabla f \cdot \nabla g}$$

$$= f\Delta g + g\Delta f + \nabla f \cdot \nabla g \quad \blacksquare$$

b.) Recall that $d\mathbf{n} = \langle -dy, dx \rangle$. Show that if $\Delta f = 0$ on a simple closed region D, then

$$\int_{\partial D} (\nabla f) \cdot d\mathbf{n} = 0,$$

where ∂D denotes the positively-oriented boundary curve of the region D.

$$\nabla f = \langle \partial_x f, \partial_y f \rangle$$

$$\nabla f \cdot d\vec{n} = \partial_x f(-dy) + \partial_y f \, dx$$

so $\displaystyle\int_{\partial D} \nabla f \cdot d\vec{n} = \int_{\partial D} \partial_y f \, dx - \partial_x f \, dy$

By Green's Theorem, this equals

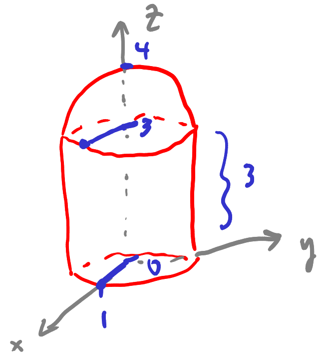
$$\iint_D \frac{\partial}{\partial x}(\partial_x f) - \frac{\partial}{\partial y}(\partial_y f) \, dA = \iint_D -\partial_x^2 f - \partial_y^2 f \ dA = \iint_D -\Delta f \ dA = \iint_D 0 \, dA = 0.$$

by the assumption that $\Delta f = 0$. Thus $\displaystyle\int_{\partial D} \nabla f \cdot d\vec{n} = 0.$

5. Consider the vector field $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$.

Use your favorite method to compute the flux, $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the surface of the solid bounded by the upper half sphere $z = 3 + \sqrt{1 - x^2 - y^2}$, the cylinder $x^2 + y^2 = 1$, and the disk $x^2 + y^2 \leq 1, z = 0$.

The surface looks like



By the Divergence Thm, Flux $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} F \, dV$.

but, $\operatorname{div} F = 1 + 1 + 1 = 3$, so $\text{flux} = 3 \cdot \iiint_E 1 \, dV = 3 \cdot \text{volume}(E)$.

But the volume can be computed without calculus:

 =  = $\frac{1}{2} \cdot \frac{4}{3} \cdot \pi \cdot 1^3 + \pi \cdot 1^2 \cdot 3 = \frac{2}{3}\pi + 3\pi$

Thus, the flux is

$$F = \iint_S \vec{F} \cdot d\vec{S} = 3\left(\frac{11}{3}\pi\right) = \boxed{11\pi}$$

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