M344: Calculus IV

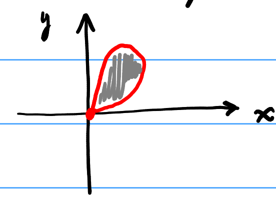
Final Exam Review Guide – Brief Solutions

1. Describe the region whose area is given by

$$\int_0^{\pi/2} \int_0^{\sin 2\theta} r \, dr \, d\theta,$$

and evaluate the integral.

Region: $r = \sin(2\theta)$, $0 \le \theta \le \pi/2$



$$\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta = \int_0^{\pi/2} \tfrac{1}{2}\sin^2(2\theta)\,d\theta$$

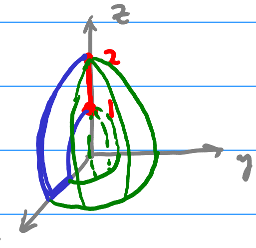
$$= \tfrac{1}{4}\int_0^{\pi/2} 1 - \cos 4\theta \, d\theta = \tfrac{1}{4}\theta - \tfrac{1}{16}\sin 4\theta \Big|_0^{\pi/2} = \pi/8$$

2. Describe the region whose volume is given by

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta,$$

and evaluate the integral.

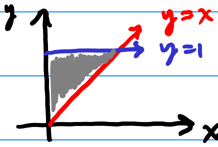
Region:



$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\varphi\,d\varphi \int_1^2 \rho^2\,d\rho = \left(\tfrac{\pi}{2}\right)(1)\left(\tfrac{7}{3}\right) = \boxed{\tfrac{7\pi}{6}}$$

3. Find the exact value of the integral, $\int_0^1 \int_x^1 \cos(y^2)\, dy\, dx.$

$y: x \to 1$
$x: 0 \to 1$

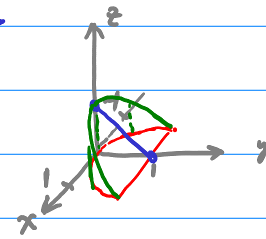


$$\int_0^1\int_x^1 \cos(y^2)\,dy\,dx = \int_0^1\int_0^y \cos(y^2)\,dx\,dy = \tfrac{1}{2}\int_0^1 2y\cos(y^2)\,dy \left\{\begin{matrix} u=y^2 & u(1)=1 \\ du = 2y\,dy & u(0)=0 \end{matrix}\right\} = \tfrac{1}{2}\sin u \Big|_0^1 = \boxed{\tfrac{1}{2}\sin(1)}$$

4. Rewrite the integral as in iterated integral in the order $dx\,dy\,dz$.

$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx.$$

$z: 0 \to 1-y$
$y: x^2 \to 1$
$x: -1 \to 1$



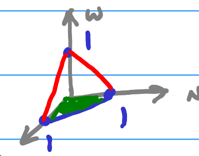
$dx\,dy\,dz$: $\begin{cases} x: -\sqrt{y} \to \sqrt{y} \\ y: 0 \to 1-z \\ z: 0 \to 1 \end{cases}$: The integral becomes

$$\int_0^1\int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz$$

5. Use the transformation $x = u^2, y = v^2, z = w^2$ to find the volume of the region bounded by the surface

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$$

and the coordinate planes.



$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{pmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{pmatrix}$ $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = 8uvw.$ $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$ becomes $u+v+w=1$

The volume is then given by $\int_0^1\int_0^{1-v}\int_0^{1-u-v} 8uvw\,dw\,du\,dv = \int_0^1\int_0^{1-v} 4uv(1-u-v)^2\,du\,dv$

$$= \int_0^1\int_0^{1-v} 4uv(1-u-v-u+u^2+uv-v+uv+v^2)\,du\,dv = 4\int_0^1\int_0^{1-v} uv - 2u^2v - 2uv^2 + 2u^2v^2 + u^3v + uv^3\,du\,dv$$

$$= 4\int_0^1 \tfrac{1}{2}u^2v - \tfrac{2}{3}u^3v - u^2v^2 + \tfrac{2}{3}u^3v^2 + \tfrac{1}{4}u^4v + \tfrac{1}{2}u^2v^3 \Big|_0^{1-v} dv$$

$$= 4\int_0^1 \tfrac{1}{2}(1-v)^2 v - \tfrac{2}{3}(1-v)^3 v - (1-v)^2 v^2 + \tfrac{2}{3}(1-v)^3 v^2 + \tfrac{1}{4}(1-v)^4 v + \tfrac{1}{2}(1-v)^2 v^3 \, dv$$

$$= 4\int_0^1 \tfrac{1}{2}(1-2v+v^2)v - \tfrac{2}{3}(1-3v+3v^2-v^3)v - (1-2v+v^2)v^2 + \tfrac{2}{3}(1-3v+3v^2-v^3)v^2 + \tfrac{1}{4}(1-4v+6v^2-4v^3+v^4)v + \tfrac{1}{2}(1-2v+v^2)v^3 \, dv$$

$$= 4\int_0^1 \cancel{\tfrac{1}{2}v} - \cancel{v^2} + \cancel{\tfrac{1}{2}v^3} - \cancel{\tfrac{2}{3}v} + \cancel{2v^2} - \cancel{2v^3} + \cancel{\tfrac{2}{3}v^4} - \cancel{v^2} + \cancel{2v^3} - \cancel{v^4} + \cancel{\tfrac{2}{3}v^2} - \cancel{2v^3} + \cancel{2v^4} - \cancel{\tfrac{2}{3}v^5} + \cancel{\tfrac{1}{4}v} - \cancel{v^2} + \cancel{\tfrac{6}{4}v^3} - \cancel{v^4} + \cancel{\tfrac{1}{4}v^5} + \cancel{\tfrac{1}{2}v^3} - \cancel{v^4} + \cancel{\tfrac{1}{2}v^5} \, dv$$

$$= 4\int_0^1 \left(\tfrac{1}{2}-\tfrac{2}{3}+\tfrac{1}{4}\right)v + \left(-1+2-1+\tfrac{2}{3}-1\right)v^2 + \left(\tfrac{1}{2}-2+2-2+\tfrac{3}{2}+\tfrac{1}{2}\right)v^3 + \left(\tfrac{2}{3}-1+2-1-1\right)v^4 + \left(-\tfrac{2}{3}+\tfrac{1}{4}+\tfrac{1}{2}\right)v^5 \, dv$$

$$= 4\int_0^1 \tfrac{1}{12}v - \tfrac{1}{3}v^2 + \tfrac{1}{2}v^3 - \tfrac{1}{3}v^4 + \tfrac{1}{12}v^5 \, dv$$

$$= 4\left(\tfrac{1}{24} - \tfrac{1}{9} + \tfrac{1}{8} - \tfrac{1}{15} + \tfrac{1}{72}\right)$$

$$= \boxed{1/90}$$ Phew!

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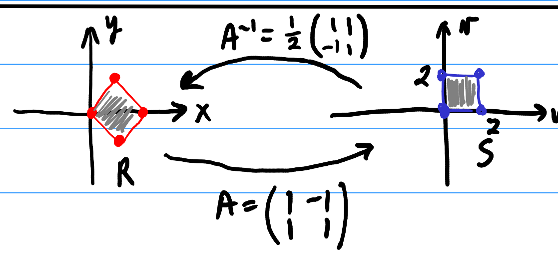
6. Use the change-of-coorindates formula and an appropriate transformation to evaluate

$$\iint_R xy\,dA$$

where R is the square with vertices $(0,0)$, $(1,1)$, $(2,0)$, and $(1,-1)$.



$\begin{cases} u = x+y \\ v = -x+y \end{cases}$ & $\begin{cases} x = \frac{1}{2}u - \frac{1}{2}v \\ y = \frac{1}{2}u + \frac{1}{2}v \end{cases}$ ← Transformation

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}, \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

So, $$\iint_R xy\,dA = \iint_S \tfrac{1}{4}(u^2-v^2)\cdot\tfrac{1}{2}\,du\,dv = \int_0^2\int_0^2 \tfrac{1}{8}(u^2-v^2)\,du\,dv = \int_0^2 \tfrac{1}{8}\left(\tfrac{1}{3}u^3 - uv^2\right)\Big|_0^2 dv$$

$$= \frac{1}{8}\int_0^2 \frac{8}{3} - 2v^2\,dv = \frac{1}{8}\left(\frac{8}{3}v - \frac{2}{3}v^3\right)\Big|_0^2 = \frac{1}{8}\left(2\cdot\frac{8}{3} - \frac{2}{3}\cdot 8\right) = 0.$$

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7. Show that $\mathbf{F}$ is a conservative vector field and use this fact to evaluate the path integral, $\int_C \mathbf{F}\cdot d\mathbf{r}$.

$$\begin{cases} \mathbf{F} = (4x^3y^2 - 2xy^3)\mathbf{i} + (2x^4y - 3x^2y^2 + 4y^3)\mathbf{j}, \\ C : \mathbf{r}(t) = (t + \sin(\pi t))\mathbf{i} + (2t + \cos(\pi t))\mathbf{j}, \quad 0 \le t \le 1. \end{cases}$$

$F = \langle P, Q\rangle$, $\frac{\partial Q}{\partial x} = 8x^3y - 6xy^2$ $\quad \frac{\partial P}{\partial y} = 8x^3y - 6xy^2$, $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Rightarrow$ Conservative!

Potential function:

$f = \int P\,dx = \int 4x^3y^2 - 2xy^3\,dx = x^4y^2 - x^2y^3 + C_1(y)$

and $f = \int Q\,dy = \int 2x^4y - 3x^2y^2 + 4y^3\,dy = x^4y^2 - x^2y^3 + y^4 + C_2(x)$

$\Big\}$ $\underline{f(x,y) = x^4y^2 - x^2y^3 + y^4}$

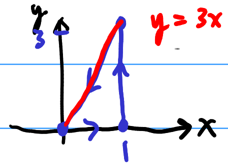
$B = \vec{r}(1) = \langle 1+0,\, 2-1\rangle = \langle 1,1\rangle \qquad A = \vec{r}(0) = \langle 0+0,\, 0+1\rangle = \langle 0,1\rangle$

by FTCPI,

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A) = (1^4\cdot1^2 - 1^2\cdot1^3 + 1^4) - (0^4\cdot1^2 - 0^2\cdot1^3 + 1^4) = 1 - 1 = \boxed{0.}$$

8. Evaluate the path integral $\int_C \sqrt{1+x^3}\,dx + 2xy\,dy$, where C is the (positively-oriented) triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$. [Hint: Use Green's Theorem.]

$\frac{\partial P}{\partial y} = 0 \qquad \frac{\partial Q}{\partial x} = 2y$



$$\int_C \sqrt{1+x^3}\,dx + 2xy\,dy = \iint_D 2y\,dy\,dx$$

$$= \int_0^1 \int_0^{3x} 2y\,dy\,dx = \int_0^1 9x^2\,dx = \boxed{3}$$

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9. Show that there is no vector field $\mathbf{G}$ satisfying $\mathrm{curl}(\mathbf{G}) = \langle 2x, 3yz, -xz^2\rangle$.

If $\mathrm{curl}\,\vec{G} = \langle 2x, 3yz, -xz^2\rangle$, then $\mathrm{div}(\mathrm{curl}\,G)$ must equal 0.

But,

$$\mathrm{div}(\langle 2x, 3yz, -xz^2\rangle) = 2 + 3z - 2xz \neq 0.$$

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10. Suppose f is a harmonic function on an open domain $D \subseteq \mathbb{R}^2$; that is, $\Delta f = 0$ on D. Show that $\int_C \partial_y f\,dx - \partial_x f\,dy$ is independent of path in D.

By Green's Theorem, $\int_{C'} \partial_y f\,dx - \partial_x f\,dy = \iint_{D'} -\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\,dA = -\iint_{D'} \Delta f\,dA = -\iint_{D'} 0\,dA = 0$

for any simple closed curve C' contained in D. Therefore $\int_C \partial_y f\,dx - \partial_x f\,dy$ is independent of path in D.

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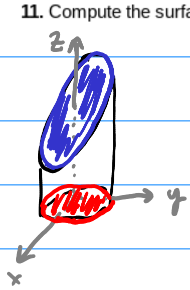
11. Compute the surface integral $\iint_S (x^2z + y^2z)\,dS$, where S is the part of the plane $z = 4 + x + y$ that lies inside of the cylinder $x^2 + y^2 = 4$.



$$\vec{r}\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 4 + r\cos\theta + r\sin\theta \end{cases} \qquad r: 0 \to 2,\quad \theta: 0 \to 2\pi.$$

$\vec{r}_r = \langle \cos\theta, \sin\theta, \cos\theta + \sin\theta\rangle$

$\vec{r}_\theta = \langle -r\sin\theta, +r\cos\theta, r\cos\theta - r\sin\theta\rangle$

$$\vec{r}_r \times \vec{r}_\theta = r\langle \cancel{\sin\theta\cos\theta}\underbrace{-\sin^2\theta - \cos^2\theta}_{-1} - \cancel{\sin\theta\cos\theta}, -\cancel{\sin\theta\cos\theta}\underbrace{-\sin^2\theta - \cos^2\theta}_{-1} + \cancel{\sin\theta\cos\theta}, \underbrace{\cos^2\theta + \sin^2\theta}_{1}\rangle$$

$$= \langle -r, -r, r\rangle$$

$\|\vec{r}_r \times \vec{r}_\theta\| = \sqrt{3}\,r$

$$x^2z + y^2z = r^2\cos^2\theta\, z + r^2\sin^2\theta\, z = r^2 z = 4r^2 + r^3\cos\theta + r^3\sin\theta$$

So $$\iint_S (x^2z + y^2z)\,dS = \int_0^{2\pi}\int_0^2 \sqrt{3}\,r\left(4r^2 + r^3\cos\theta + r^3\sin\theta\right)dr\,d\theta$$

$$= \sqrt{3}\int_0^{2\pi} r^4 + \frac{1}{5}r^5\cos\theta + \frac{1}{5}r^5\sin\theta\Big|_0^2\, d\theta = \sqrt{3}\int_0^{2\pi} 16 + \frac{32}{5}\underbrace{\cancel{(\cos\theta + \sin\theta)}}_{0}\, d\theta$$

$$= \boxed{32\sqrt{3}\,\pi}$$

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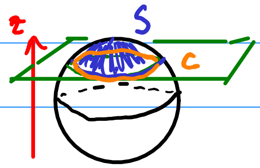
12. Evaluate $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F} = \langle x^2yz, yz^2, z^3e^{xy}\rangle$ and S is the part of the sphere $x^2+y^2+z^2=5$ that lies above the plane $z=1$. Take S to be oriented upward. [Hint: Use Stokes' Theorem.]



Boundary: $x^2+y^2=4$, $z=1$: $\left\{ \vec{r}_c = \langle 2\cos\theta, 2\sin\theta, 1\rangle \right\}$

$\langle -y, x, 0\rangle$

Stokes Theorem: $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle x^2yz, yz^2, z^3e^{xy}\rangle\cdot\langle -2\sin\theta, 2\cos\theta, 0\rangle\, d\theta$

$$= \int_0^{2\pi} -x^2y^2z + xyz + 0\; d\theta = \int_0^{2\pi} -16\cos^2\theta\sin^2\theta + 4\cos\theta\sin\theta\; d\theta$$

$$= -4\int_0^{2\pi} 1-\cos^2(2\theta)\, d\theta + 4\int_0^{2\pi}\cos\theta\sin\theta\, d\theta$$

$$= -4\int_0^{2\pi} 1\, d\theta + 2\int_0^{2\pi} 1 + \underbrace{\cancel{\cos(4\theta)}}_{=0}\, d\theta + 4\underbrace{\cancel{\int_0^{2\pi}\cos\theta\sin\theta\, d\theta}}_{=0}$$

$u = \sin\theta$ $\quad u(2\pi) = 0$

$du = \cos\theta\, d\theta$ $\quad u(0) = 0$

$$= -8\pi + 4\pi = \boxed{-4\pi}$$

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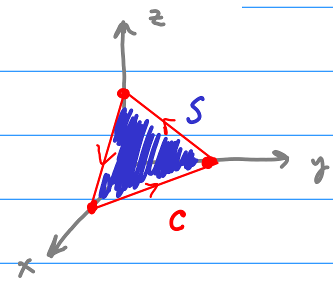
13. Evaluate the path integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle xy, yz, zx\rangle$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, oriented counter-clockwise when looking "from above." [Hint: Use Stokes' Theorem.]



S is the graph of $z = 1-x-y$ over the triangle bounded by $y = 1-x$ and the coordinate axes.

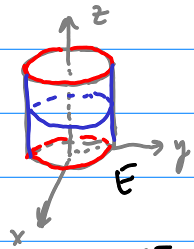
$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ xy & yz & zx \end{vmatrix} = \langle 0-y, 0-z, 0-x\rangle = -\langle y, z, x\rangle$$

$$S:\ \left.\begin{aligned} \vec{r} &= \langle x, y, 1-x-y\rangle \\ \vec{r}_x &= \langle 1, 0, -1\rangle \\ \vec{r}_y &= \langle 0, 1, -1\rangle \end{aligned}\right\}\quad \vec{r}_x\times\vec{r}_y = \langle 1,1,1\rangle$$

So, $\int_C \vec{F}\cdot d\vec{r} = \iint_D -\langle y, z, x\rangle\cdot\langle 1,1,1\rangle\, dA = \int_0^1\int_0^{1-x} \cancel{-x}\,\cancel{-y} - (1\cancel{-x}\cancel{-y})\; dy\, dx$

$$= -\int_0^1 (1-x)\,dx = -\left(x - \tfrac{1}{2}x^2\right)\Big|_0^1 = -\left(1-\tfrac{1}{2}\right) = \boxed{-\tfrac{1}{2}}$$

14. Use the Divergence Theorem to calculate the flux of $\mathbf{F}$ across the surface S, where $\mathbf{F} = \langle x^3, y^3, z^3 \rangle$ and S is the surface bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2$.



$$\text{div}\,\vec{F} = 3x^2 + 3y^2 + 3z^2$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \text{div}\,\vec{F}\,dV = \int_0^{2\pi}\int_0^1\int_0^2 3(r^2+z^2)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1\int_0^2 3r^3 + 3rz^2\,dz\,dr = 2\pi \int_0^1 6r^3 + 8r\,dr = 2\pi\left(6/4 + 4\right) = \boxed{\frac{22\pi}{4}}$$

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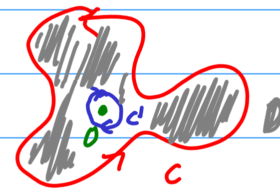
15. Let

$$\mathbf{F}(x,y) = \frac{(2x^3 + 2xy^2 - 2y)\mathbf{i} + (2y^3 + 2x^2y + 2x)\mathbf{j}}{x^2 + y^2}.$$

Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is any positively-oriented Jordan curve that encloses the origin.



Let C be any positively-oriented curve enclosing $\vec{0}$, and let C' be a negatively-oriented circle contained in C.

Then Green's Theorem applies in D.

$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)(4xy+2) - (2y^3+2x^2y+2x)(2x)}{(x^2+y^2)^2} = \frac{\cancel{4x^3y} + 2x^2 + \cancel{4xy^3} + 2y^2 - (\cancel{4xy^3} + \cancel{4x^3y} + 4x^2)}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial y} = \frac{(x^2+y^2)(4xy-2) - (2x^3+2xy^2-2y)(2y)}{(x^2+y^2)^2} = \frac{\cancel{4x^3y} - 2x^2 + \cancel{4xy^3} - 2y^2 - (\cancel{4x^3y} + \cancel{4xy^3} - 4y^2)}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

so $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$ and $\displaystyle\int_C \vec{F}\cdot d\vec{r} + \int_{C'} \vec{F}\cdot d\vec{r} = 0$

This implies that $\int_C \vec{F}\cdot d\vec{r} = \int_u \vec{F}\cdot d\vec{r}$ where u is the positively-oriented unit circle. (Compare w/ the exercise we did together in class.)

u: $x = \cos\theta$
$y = \sin\theta$

$$\vec{F}(\theta) = \left\langle 2\cos^3\theta + 2\cos\theta\sin^2\theta - 2\sin\theta,\ 2\sin^3\theta + 2\cos^2\theta\sin\theta + 2\cos\theta \right\rangle$$

$$d\vec{r} = \langle -\sin\theta, \cos\theta \rangle\,d\theta$$

$$\int_u \vec{F}\cdot d\vec{r} = \int_0^{2\pi} -2\cos^3\theta\sin\theta - 2\cos\theta\sin^3\theta + 2\sin^2\theta + 2\sin^3\theta\cos\theta + 2\cos^3\theta\sin\theta + 2\cos^2\theta\,d\theta$$

$$= \int_0^{2\pi} 2\,d\theta = \boxed{4\pi}$$