M243 - Calc III

Midterm Exam Review – Brief Solutions

1. Find an equation of the plane through the points $P(1,2,3)$, $Q(4,0,-1)$, and $R(2,-4,-2)$.

$\vec{a} = \vec{PQ} = \langle 3,-2,-4\rangle$ $\quad \vec{n} = \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & -4 \\ 1 & -6 & -5 \end{vmatrix} = \langle 10-24, -4+15, -18+2\rangle = \langle -14, 11, -16\rangle$

$\vec{b} = \vec{PR} = \langle 1,-6,-5\rangle$

$\vec{r}_0 = \langle 1,2,3\rangle$

$d = -\vec{n}\cdot\vec{r}_0 = -\langle 1,2,3\rangle\cdot\langle -14,11,-16\rangle = -(-14+22-48) = -(-40) = 40$

$$\boxed{\Pi:\ -14x + 11y - 16z + 40 = 0}$$

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2. Find the distance between the point and the given plane.

$$\begin{cases} P(1,-3,2) \\ \Pi:\ 3x+2y+6z=5 \end{cases}$$

$\vec{r}_0 = \langle 1,-3,2\rangle$

$\vec{n} = \langle 3,2,6\rangle$

$$D = \frac{|d+\vec{n}\cdot\vec{r}_0|}{\|\vec{n}\|} = \frac{|-5+(3-6+12)|}{\sqrt{9+4+36}} = \frac{|-5+9|}{7} = \frac{4}{7}.$$

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3. Reduce the equation to one of the standard forms and classify the surface.

$$x^2 - y^2 - z^2 - 4x - 2z + 3 = 0$$

$x^2 - 4x + 4 \quad - y^2 \quad - (z^2 + 2z + 1) = -3 + 4 - 1$

$(x-2)^2 - y^2 - (z+1)^2 = 0$

$(x-2)^2 = y^2 + (z+1)^2$ $\qquad$ This is a <u>CONE</u>.

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4. You must be able to match the graph of a surface in $\mathbb{R}^3$ to its equation. (This will be a multiple choice question.)

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5. Consider the vector function

$$\mathbf{r}(t) = \frac{t^2-1}{t-1}\mathbf{i} + \sqrt{t+8}\,\mathbf{j} + \frac{\sin \pi t}{\ln t}\mathbf{k}.$$

a.) What is the domain of $\mathbf{r}$?

b.) Compute $\lim_{t\to 1} \mathbf{r}(t)$, provided it exists.

c.) Compute $\dot{\mathbf{r}}(t)$, provided it exists.

a) $\begin{cases} x(t) = \dfrac{t^2-1}{t-1} \\ y(t) = \sqrt{t+8} \\ z(t) = \dfrac{\sin(\pi t)}{\ln t} \end{cases}$ $\quad$ dom(x): $t \neq 1$

dom(y): $t+8 \geq 0 \Rightarrow t \geq -8$

dom(z): $t > 0$

dom$(\vec{r}) = (0,1)\cup(1,\infty)$

b) $\lim_{t\to 1} \vec{r}(t) = \lim_{t\to 1} \left\langle \frac{t^2-1}{t-1}, \sqrt{t+8}, \frac{\sin(\pi t)}{\ln t} \right\rangle = \left\langle \lim_{t\to 1} \frac{(t+1)\cancel{(t-1)}}{\cancel{t-1}}, \lim_{t\to 1} \sqrt{t+8}, \lim_{t\to 1} \frac{\sin(\pi t)}{\ln t} \right\rangle$

$= \left\langle 2, 3, \lim_{t\to 1} \frac{\pi \cos(\pi t)}{1/t} \right\rangle = \langle 2, 3, -\pi/1 \rangle = \langle 2, 3, -\pi \rangle$

c) $\begin{cases} x(t) = \frac{t^2-1}{t-1} = t+1, \quad t \neq 1 \\ y(t) = \sqrt{t+8} \\ z(t) = \frac{\sin(\pi t)}{\ln t} \end{cases}$ $\begin{cases} \dot{x}(t) = 1, \quad t \neq 1 \\ \dot{y}(t) = \frac{1}{2\sqrt{t+8}} \\ \dot{z}(t) = \frac{\ln t \cdot \pi \cos(\pi t) - 1/t \sin(\pi t)}{(\ln t)^2} = \frac{\pi t \ln t \cos(\pi t) - \sin(\pi t)}{t (\ln t)^2} \end{cases}$

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6. Find a parametrization of the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the surface $z = xy$.

(Recall that "parametrization" is another name for a vector function whose terminal points trace out the space curve.)

$x^2+y^2=4: \quad \begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \end{cases}$ $\quad z = xy = 2\cos\theta \cdot 2\sin\theta = 4\cos\theta\sin\theta = 2\sin(2\theta).$

so, $\vec{r}(t) = \langle 2\cos\theta, 2\sin\theta, 2\sin(2\theta) \rangle, \quad 0 \le \theta \le 2\pi.$

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7. Let $\mathbf{u}$ and $\mathbf{v}$ be vector functions in $\mathbb{R}^3$. Prove the product rule for the dot product:

$$\frac{d}{dt}\Big[\mathbf{u}(t) \cdot \mathbf{v}(t)\Big] = \dot{\mathbf{u}}(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \dot{\mathbf{v}}(t).$$

Hint. Write both $\mathbf{u}$ and $\mathbf{v}$ in coordinates.

$\vec{u}(t) = \langle u_1(t), u_2(t), u_3(t) \rangle \qquad \dot{\vec{u}}(t) = \langle \dot{u}_1(t), \dot{u}_2(t), \dot{u}_3(t) \rangle$

$\vec{v}(t) = \langle v_1(t), v_2(t), v_3(t) \rangle \qquad \dot{\vec{v}}(t) = \langle \dot{v}_1(t), \dot{v}_2(t), \dot{v}_3(t) \rangle$

$$\begin{aligned} \dot{\vec{u}} \cdot \vec{v} + \vec{u} \cdot \dot{\vec{v}} &= \langle \dot{u}_1, \dot{u}_2, \dot{u}_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle + \langle u_1, u_2, u_3 \rangle \cdot \langle \dot{v}_1, \dot{v}_2, \dot{v}_3 \rangle \\ &= \underline{\dot{u}_1 v_1} + \underline{\dot{u}_2 v_2} + \underline{\dot{u}_3 v_3} + \underline{u_1 \dot{v}_1} + \underline{u_2 \dot{v}_2} + \underline{u_3 \dot{v}_3} \\ &= (\dot{u}_1 v_1 + u_1 \dot{v}_1) + (\dot{u}_2 v_2 + u_2 \dot{v}_2) + (\dot{u}_3 v_3 + u_3 \dot{v}_3) \\ &= (u_1 v_1)^{\cdot} + (u_2 v_2)^{\cdot} + (u_3 v_3)^{\cdot} \\ &= d/dt \left[u_1 v_1 + u_2 v_2 + u_3 v_3 \right] \\ &= \frac{d}{dt} \left[\vec{u} \cdot \vec{v} \right]. \quad \square \end{aligned}$$

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8. Consider the vector function $\mathbf{r}(t) = \langle \arctan(t), 2e^{2t}, 8te^t \rangle$. Find the unit tangent vector $\mathbf{T}(0)$.

$\dot{\vec{r}}(t) = \left\langle \frac{1}{1+t^2}, 4e^{2t}, 8e^t + 8te^t \right\rangle$

$\dot{\vec{r}}(0) = \langle 1, 4, 8 \rangle$

$\|\dot{\vec{r}}(0)\| = \sqrt{1+16+64} = \sqrt{81} = 9$

$\vec{T}(0) = \frac{\dot{\vec{r}}(0)}{\|\dot{\vec{r}}(0)\|} = \langle 1/9, 4/9, 8/9 \rangle$

9. Find $\mathbf{r}(t)$ if $\dot{\mathbf{r}}(t) = \langle t, e^t, te^t \rangle$ and $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

$$\int \dot{\vec{r}}(t)\,dt = \langle \tfrac{1}{2}t^2 + C_1, e^t + C_2, te^t - e^t + C_3 \rangle = \vec{r}(t)$$

$$\vec{r}(0) = \langle C_1, 1 + C_2, -1 + C_3 \rangle = \langle 1, 1, 1 \rangle \Rightarrow \vec{C} = \langle 1, 0, 2 \rangle$$

so,

$$\vec{r}(t) = \langle \tfrac{1}{2}t^2 + 1, e^t, te^t - e^t + 2 \rangle$$

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10. Reparametrize the plane curve

$$\mathbf{r}(t) = \left(\frac{2}{t^2-1} - 1\right)\mathbf{i} + \frac{2t}{t^2+1}\mathbf{j}$$

with respect to arc length from the point $(1, 0)$ in the direction of increasing t. Express the reparametrization in simplest form. What can you deduce about the curve?

$$s(t) = \int_0^t \|\dot{\vec{r}}(u)\|\,du$$

$$\dot{\vec{r}}(t) = \left\langle -4t(t^2-1)^{-2}, \frac{(t^2+1)2 - 2t(2t)}{(t^2+1)^2} \right\rangle$$

$$\dot{\vec{r}}(t) = \left\langle \frac{-4t}{(t^2-1)^2}, \frac{2-2t^2}{(t^2+1)^2} \right\rangle$$

$$\|\dot{\vec{r}}(t)\| = \sqrt{\frac{16t^2}{(t^2-1)^4} + \frac{(2-2t^2)^2}{(t^2+1)^4}} = \sqrt{\frac{(t^2+1)^4 16t^2 + (2-2t^2)^2(t^2-1)^4}{(t^4-1)^4}}$$

$$= \sqrt{\frac{\quad}{(t^4-1)^2}}$$

Hmm, I think there is a typo in this question?

Idea: Find $s = s(t)$, solve for t, plug in t as a function of s, reduce.

11. Compute the curvature κ of the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at the point $P(1,1,1)$.

$$\dot{\vec r} = \langle 1, 2t, 3t^2 \rangle \qquad \dot{\vec r}(1) = \langle 1,2,3 \rangle \qquad \dot{\vec r} \times \ddot{\vec r}(1) = \langle 6, -6, 2 \rangle$$

$$\ddot{\vec r} = \langle 0, 2, 6t \rangle \qquad \ddot{\vec r}(1) = \langle 0,2,6 \rangle \qquad \|\dot{\vec r} \times \ddot{\vec r}(1)\| = \sqrt{36+36+4} = \sqrt{76}$$

$$\|\dot{\vec r}(1)\| = \sqrt{1+4+9} = \sqrt{14}$$

$$\kappa(1) = \frac{\|\dot{\vec r} \times \ddot{\vec r}\|}{\|\dot{\vec r}\|^3}(1) = \frac{\sqrt{76}}{14\sqrt{14}}$$

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12. At what point on the curve $x = t^3$, $y = 3t$, $z = t^4$ is the normal plane parallel to the plane $6x + 6y - 8z = 1$?

Hint. The normal plane has normal vector $\mathbf{T}$: it is spanned by the normal and binormal vectors, $\mathbf{N}$ and $\mathbf{B}$.

$$\vec r(t) = \langle t^3, 3t, t^4 \rangle, \quad \dot{\vec r}(t) = \langle 3t^2, 3, 4t^3 \rangle = \lambda \langle 6, 6, -8 \rangle$$

$$\begin{cases} 3t^2 = 6\lambda & \Rightarrow \quad 3t^2 = 3 \quad \Rightarrow \quad t = \cancel{\pm}1 = -1 \\ 3 = 6\lambda & \Rightarrow \quad \lambda = 1/2 \\ 4t^3 = -8\lambda & \Rightarrow \quad 4t^3 = -4 \quad \Rightarrow \quad t = -1 \end{cases}$$

So the point is $P = \vec r(-1) = (3, 3, -4)$.

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13. Show that the osculating plane at every point on the curve $\mathbf{r}(t) = \left\langle t+2, 1-t, \frac{1}{2}t^2 \right\rangle$ is the same plane. What can you conclude about the curve?

$$\dot{\vec r}(t) = \langle 1, -1, 2t \rangle \qquad \|\dot{\vec r}(t)\| = \sqrt{1+1+4t^2} = \sqrt{2+4t^2}$$

$$\vec T(t) = \left\langle \frac{1}{\sqrt{2+4t^2}}, \frac{-1}{\sqrt{2+4t^2}}, \frac{2t}{\sqrt{2+4t^2}} \right\rangle = \frac{1}{\sqrt{2}} \left\langle \frac{1}{\sqrt{2t^2+1}}, \frac{-1}{\sqrt{2t^2+1}}, \frac{2t}{\sqrt{2t^2+1}} \right\rangle$$

$$\dot{\vec T}(t) = \left\langle \frac{-\frac{1}{2}\cdot 8t}{\sqrt{2+4t^2}^{\,3}}, \frac{\frac{1}{2}\cdot 8t}{\sqrt{2+4t^2}^{\,3}}, \frac{\sqrt{2+4t^2}\cdot 2 - \cancel{2}t \cdot \frac{8t}{\cancel{2}\sqrt{2+4t^2}}}{\left(\sqrt{2+4t^2}\right)^2} \right\rangle = \left\langle \frac{-4t}{\sqrt{2+4t^2}^{\,3}}, \frac{4t}{\sqrt{2+4t^2}^{\,3}}, \frac{(2+4t^2)\cdot 2 - 8t^2}{\sqrt{2+4t^2}^{\,3}} \right\rangle$$

$$= \cancel{\frac{1}{\sqrt{2+4t^2}^{\,3}}} \underbrace{\langle -4t, 4t, 4 \rangle} = \dot{\vec{\mathbb{T}}}$$

$$\|\dot{\vec{\mathbb{T}}}\| = \sqrt{16t^2 + 16t^2 + 16} = 4\sqrt{2t^2+1}$$

So,

$$\vec N = \left\langle \frac{-t}{\sqrt{2t^2+1}}, \frac{t}{\sqrt{2t^2+1}}, \frac{1}{\sqrt{2t^2+1}} \right\rangle$$

$$\vec T \times \vec N = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2t^2+1}} \right)^2 \begin{vmatrix} \vec i & \vec j & \vec k \\ 1 & -1 & 2t \\ -t & t & 1 \end{vmatrix} = \frac{1}{\sqrt{2}(2t^2+1)} \langle -1-2t^2, -2t^2-1, t-t \rangle$$

$$= \frac{1}{\sqrt{2}}\langle -1,-1,0\rangle = \left\langle \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right\rangle = \vec{B}$$

Since $\vec{B}$ is constant, then the osculating plane is constant and the curve lies in a single plane.

14. Let C be a smooth space curve with unit tangent vector field $\mathbf{T}$. Prove that $\mathbf{T} \perp \dot{\mathbf{T}}$ for all t in the domain of $\mathbf{T}$.

$$\|\vec{T}\| = 1 \Rightarrow \vec{T}\cdot\vec{T} = 1$$

$$\text{Then, } \frac{d}{dt}\left(\vec{T}\cdot\vec{T} = 1\right)$$

$$\Rightarrow \frac{d}{dt}(\vec{T}\cdot\vec{T}) = 0$$

$$\Rightarrow \dot{\vec{T}}\cdot\vec{T} + \vec{T}\cdot\dot{\vec{T}} = 0$$

$$\Rightarrow 2\dot{\vec{T}}\cdot\vec{T} = 0$$

$$\Rightarrow \dot{\vec{T}}\cdot\vec{T} = 0$$

$$\Rightarrow \vec{T} \perp \dot{\vec{T}} \quad \square$$

15. Prove that the curvature of a circle of radius a is constant, $\kappa = 1/a$.

$$\vec{r}(t) = \langle a\cos t, a\sin t\rangle$$

$$\dot{\vec{r}}(t) = \langle -a\sin t, a\cos t\rangle$$

$$\ddot{\vec{r}}(t) = \langle -a\cos t, -a\sin t\rangle$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \langle 0, 0, a^2\sin^2 t + a^2\cos^2 t\rangle = \langle 0,0,a^2\rangle$$

$$\|\dot{\vec{r}} \times \ddot{\vec{r}}\| = \sqrt{(a^2)^2} = a^2$$

$$\|\dot{\vec{r}}\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = a$$

$$\text{So, } \kappa(t) = \frac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|^3} = \frac{a^2}{a^3} = \frac{1}{a} \quad \blacksquare$$

16. Show that the limit does not exist.

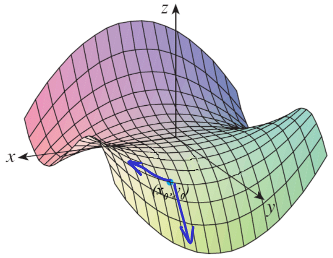
$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 30y^2}{x^2 + 15y^2}$$

put $x=0$: $\displaystyle\lim_{y\to 0} \frac{-30y^2}{15y^2} = -2$

$y=0$: $\displaystyle\lim_{x\to 0} \frac{x^4}{x^2} = \lim_{x\to 0} x^2 = 0.$

These do not coincide, so the limit does not exist

17. Determine the signs of the partial derivatives for the function f whose graph is show, at the point (x_0, y_0).



$$\frac{\partial f}{\partial x}(x_0,y_0) > 0 \qquad \frac{\partial f}{\partial y}(x_0,y_0) < 0$$

18. Compute the second directional derivative $D^2_{\mathbf{v}}f(x,y) = D_{\mathbf{v}}[D_{\mathbf{v}}f(x,y)]$ for $f(x,y) = x^3 + 5x^2y + y^3$ in the direction of the vector $\mathbf{v} = \langle 3, 4\rangle$ and evaluate it at the point $(3,2)$.

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \langle 3/5, 4/5\rangle \qquad D_{\vec{v}}f = \nabla f \cdot \vec{u}$$

$$\nabla f = \langle 3x^2 + 10xy, 5x^2 + 3y^2\rangle$$

$$D_{\vec{v}}f = \langle 3x^2+10xy, 5x^2+3y^2\rangle \cdot \langle 3/5, 4/5\rangle$$

$$= 9/5\, x^2 + 6xy + 4x^2 + \frac{12}{5}y^2 = 29/5\, x^2 + 6xy + 12/5\, y^2$$

$$D_{\vec{v}}[D_{\vec{v}}f] = \nabla[D_{\vec{v}}f] \cdot \vec{u}$$

$$\nabla[D_{\vec{v}}f] = \left\langle \frac{58}{5}x + 6y, 6x + \frac{24}{5}y \right\rangle$$

$$\nabla[D_{\vec{v}}f] \cdot \vec{u} = \left\langle \frac{58}{5}x + 6y, 6x + \frac{24}{5}y \right\rangle \cdot \langle 3/5, 4/5\rangle$$

$$= \frac{174}{25}x + \frac{18}{5}y + \frac{24}{5}x + \frac{96}{25}y$$

$$= \frac{294}{25}x + \frac{186}{25}y \Big|_{(x,y)=(3,2)}$$

$$= \frac{1}{25}(888 + 372) = \frac{1260}{25} = \frac{252}{5}$$

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19. Consider the function $f(x,y) = xy$ at the point $P(6,5)$, and consider the level curve $f(x,y) = 30$. Find an equation of the tangent line to the level curve at P, and compute the gradient of f at P. Then show that the gradient vector is perpendicular to the level curve at P.

$$\nabla f = \langle y, x\rangle \qquad \nabla f(P) = \langle 5, 6\rangle$$

$$f(x,y) = xy = 30 \qquad \frac{dy}{dx} = \frac{-\partial_x F(P)}{\partial_y F(P)} = \frac{-y}{x}\Big|_P = -\frac{5}{6} = \frac{dy}{dx}$$

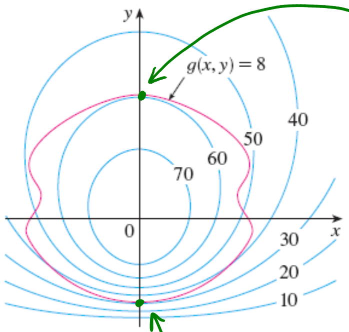
so $y = 5 - \frac{5}{6}(x-6) = -\frac{5}{6}x + 10$ is the tangent line

its direction vector is $\langle dx, dy\rangle = \langle 6, -5\rangle$

$$\nabla f \cdot \langle dx, dy\rangle = \langle 5, 6\rangle \cdot \langle 6, -5\rangle = 30 - 30 = 0.$$

so $\nabla f \perp$ the level curve!

20. In the figure below, the blue level curves correspond to a function $f(x, y)$ and the red level curve corresponds to a constraint $g(x, y) = 8$. Estimate the maximum and minimum values of f subject to the constraint.



$f(x,y) \approx 59$ Max

$f(x,y) \approx 30$ Min

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21. Find the extreme values of the function $f(x, y) = x^2 + y^2 + 4x - 4y$ on the region $x^2 + y^2 \le 81$. [Hint: Use the method of Lagrange multipliers on the boundary, and use the Second Derivative Test to check all critical points in the interior.]

Interior:

$\nabla f = \langle 2x+4, 2y-4 \rangle = \langle 0, 0 \rangle$

$\begin{cases} 2x+4=0 \\ 2y-4=0 \end{cases}$ critical point: $P(-2,2)$ $\quad f(-2,2) = (-2)^2 + 2^2 + 4(-2) - 4(2) = 4+4-8-8 = -8$

Boundary:

$g(x,y) = x^2+y^2 \qquad \nabla g = \langle 2x, 2y \rangle$

$\begin{cases} 2x+4 = 2\lambda x \\ 2y-4 = 2\lambda y \\ x^2+y^2 = 81 \end{cases} \Rightarrow \lambda = \dfrac{2x+4}{2x} = \dfrac{2y-4}{2y}$

$\Rightarrow \cancel{4xy} + 8y = \cancel{4xy} - 8x \rightarrow x = -y$

$2x^2 = 81 \Rightarrow x = \pm \dfrac{9}{\sqrt{2}}, \ y = \mp \dfrac{9}{\sqrt{2}}$

$f\left(\frac{9}{\sqrt{2}}, \frac{-9}{\sqrt{2}}\right) = \left(\frac{9}{\sqrt{2}}\right)^2 + \left(\frac{-9}{\sqrt{2}}\right)^2 + 4\frac{9}{\sqrt{2}} - 4\left(\frac{-9}{\sqrt{2}}\right) = 81 + \frac{76}{\sqrt{2}}$

$f\left(\frac{-9}{\sqrt{2}}, \frac{9}{\sqrt{2}}\right) = \left(\frac{-9}{\sqrt{2}}\right)^2 + \left(\frac{9}{\sqrt{2}}\right)^2 + 4\left(\frac{-9}{\sqrt{2}}\right) - 4\left(\frac{9}{\sqrt{2}}\right) = 81 - \frac{76}{\sqrt{2}}$

So the absolute max is $81 + \frac{76}{\sqrt{2}}$

and the absolute min is -8

22. Prove the theorem: *Suppose f is a differentiable function of at least 2 variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $\|\nabla f(\mathbf{x})\|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.*

Proof. Let θ be the angle between $\nabla f(\vec{x})$ and $\vec{u}$.

Then $D_{\vec{u}} f(\vec{x}) = \|\nabla f(\vec{x})\| \, \|\vec{u}\| \cos\theta$

$= \|\nabla f(\vec{x})\| \cos\theta$.

This will be maximum when $\cos\theta = 1$, hence $\theta = 0$. This occurs when $\vec{u}$ is in the same direction as $\nabla f(\vec{x})$.

The maximum value is $\|\nabla f(\vec{x})\|$. ∎

23. Find an equation of the osculating circle to the curve $y = 4x^2 - x^4$ at $x = 2$.

$$\kappa(x) = \frac{|f''(x)|}{\sqrt{1+f'(x)^2}^{\,3}}$$

$f(x) = 4x^2 - x^4$ $\quad$ $f(2) = 4(2^2) - 2^4 = 16 - 16 = 0$

$f'(x) = 8x - 4x^3$ $\quad$ $f'(2) = 16 - 4 \cdot 8 = -16$

$f''(x) = 8 - 12x^2$ $\quad$ $f''(2) = 8 - 12(4) = -40$

Normal line at $x=2$ has slope: $\frac{1}{16}$ $\qquad$ $\kappa(2) = \frac{|-40|}{\sqrt{1+16^2}^{\,3}} = \frac{40}{\sqrt{257}^{\,3}}$

Equation of circle: $(\cancel{x}-h)^2 + (y-k)^2 = \frac{1}{\kappa^2}$

Center of circle lies on $y = \frac{1}{16}(x-2)$: $\quad (2-x)^2 + \left(0 - \frac{1}{16}(x-2)\right)^2 = \underbrace{\frac{257^3}{40^2}}_{\approx 10609.12}$

$$\left(1 + \frac{1}{256}\right)(x-2)^2 = \frac{257^3}{40^2}$$

$$(x-2)^2 = \frac{256}{\cancel{257}} \cdot \frac{257^{\cancel{3}\,2}}{40}$$

$$x = 2 \pm \frac{16 \cdot 257}{2\sqrt{10}} = 2 - \frac{16(257)}{2\sqrt{10}} \approx -648.16$$

$$y = \frac{1}{16}\left(\cancel{2} - \frac{16(257)}{2\sqrt{10}} - \cancel{2}\right) = -\frac{257}{2\sqrt{10}} \approx -40.64$$

So the circle is

$$\boxed{(x+648.16)^2 + (y+40.64)^2 = 10609.12}$$

24. The *involute* of a space curve $\mathbf{r}(t)$ is the space curve defined by $\mathcal{I}(t) = \mathbf{r}(t) - s(t)\mathbf{T}(t)$. Show that the involute of the helix, $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ lies in the xy-plane.

$$s(t) = \int_0^t \|\dot{\vec{r}}(u)\|\,du \quad , \quad \dot{\vec{r}}(t) = \langle -\sin t, \cos t, 1\rangle \quad , \quad \|\dot{\vec{r}}(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$= \int_0^t \sqrt{2}\,du = \sqrt{2}\,t \qquad \vec{T}(t) = \frac{\dot{\vec{r}}(t)}{\|\dot{\vec{r}}(t)\|} = \left\langle -\tfrac{1}{\sqrt{2}}\sin t, \tfrac{1}{\sqrt{2}}\cos t, \tfrac{1}{\sqrt{2}}\right\rangle$$

$$\mathcal{I}(t) = \vec{r}(t) - s(t)\vec{T}(t) = \langle \cos t, \sin t, t\rangle - \sqrt{2}\,t\left\langle -\tfrac{1}{\sqrt{2}}\sin t, \tfrac{1}{\sqrt{2}}\cos t, \tfrac{1}{\sqrt{2}}\right\rangle$$

$$= \langle \cos t + t\sin t,\ \sin t - t\cos t,\ 0\rangle$$

So the involute is always in the xy-plane.

25. Let $f = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, be a smooth function satisying

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0.$$

Show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\right)$$

$$= \left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial r}\right)\frac{\partial x}{\partial r} + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial r^2} + \left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial r}\right)\frac{\partial y}{\partial r} + \frac{\partial f}{\partial y}\left(\frac{\partial^2 y}{\partial r^2}\right)$$

$$= \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial r}\right)^2 + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial r}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial r^2} + \frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial r}\frac{\partial y}{\partial r} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial r}\right)^2 + \frac{\partial f}{\partial y}\left(\frac{\partial^2 y}{\partial r^2}\right)$$

Similarly,

$$\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial \theta}\right)^2 + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial \theta}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial \theta^2} + \frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \theta} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial \theta}\right)^2 + \frac{\partial f}{\partial y}\left(\frac{\partial^2 y}{\partial \theta^2}\right)$$

Now, $\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial^2 x}{\partial r^2} = 0 \qquad \frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial^2 y}{\partial r^2} = 0$

$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial^2 x}{\partial \theta^2} = -r\cos\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta \qquad \frac{\partial^2 y}{\partial \theta^2} = -r\sin\theta$

Plugging in:

$$\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial^2 f}{\partial x^2}\overset{\cos^2\theta}{\left(\frac{\partial x}{\partial r}\right)^2} + \frac{\partial^2 f}{\partial y\partial x}\overset{\sin\theta\cos\theta}{\frac{\partial y}{\partial r}\frac{\partial x}{\partial r}} + \frac{\partial f}{\partial x}\overset{0}{\frac{\partial^2 x}{\partial r^2}} + \frac{\partial^2 f}{\partial x\partial y}\overset{\sin\theta\cos\theta}{\frac{\partial x}{\partial r}\frac{\partial y}{\partial r}} + \frac{\partial^2 f}{\partial y^2}\overset{\sin^2\theta}{\left(\frac{\partial y}{\partial r}\right)^2} + \frac{\partial f}{\partial y}\overset{0}{\left(\frac{\partial^2 y}{\partial r^2}\right)}$$

$$+\frac{1}{r}\left(\frac{\partial f}{\partial x}\underset{\cos\theta}{\frac{\partial x}{\partial r}} + \frac{\partial f}{\partial y}\underset{\sin\theta}{\frac{\partial y}{\partial r}}\right)$$

$$+\frac{1}{r^2}\left(\frac{\partial^2 f}{\partial x^2}\underset{r^2\sin^2\theta}{\left(\frac{\partial x}{\partial \theta}\right)^2} + \frac{\partial^2 f}{\partial y\partial x}\underset{-r^2\sin\theta\cos\theta}{\frac{\partial y}{\partial \theta}\frac{\partial x}{\partial \theta}} + \frac{\partial f}{\partial x}\underset{-r\cos\theta}{\frac{\partial^2 x}{\partial \theta^2}} + \frac{\partial^2 f}{\partial x\partial y}\underset{-r^2\sin\theta\cos\theta}{\frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \theta}} + \frac{\partial^2 f}{\partial y^2}\underset{r^2\cos^2\theta}{\left(\frac{\partial y}{\partial \theta}\right)^2} + \frac{\partial f}{\partial y}\underset{-r\sin\theta}{\left(\frac{\partial^2 y}{\partial \theta^2}\right)}\right)$$

$$= \underbrace{(\cos^2\theta+\sin^2\theta)}_{1}\frac{\partial^2 f}{\partial x^2} + \underbrace{(\cos^2\theta+\sin^2\theta)}_{1}\frac{\partial^2 f}{\partial y^2} + \underbrace{\left(2\sin\theta\cos\theta - \frac{1}{r^2}(2r^2\sin\theta\cos\theta)\right)}_{0}\frac{\partial^2 f}{\partial x\partial y} + \underbrace{\left(\frac{1}{r}\cos\theta - \frac{1}{r^2}r\cos\theta\right)}_{0}\frac{\partial f}{\partial x}$$

$$+\underbrace{\left(\frac{1}{r}\sin\theta - \frac{1}{r^2}r\sin\theta\right)}_{0}\frac{\partial f}{\partial y}$$

so, $$\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2} = 0 \quad \blacksquare$$