Name: Key

M344: Calculus III (Fall 2018)

Midterm Exam

Read and follow all instructions. You may not use any electronic devices, but you may use one 3×5 in^2 index card of your own hand-written notes.

Instructions

Complete all problems, showing enough work. Partial credit will be given when deserved. Clearly mark your final answers, when appropriate.

1. Find an equation of the plane through the points $P(1,2,3)$, $Q(-1,0,1)$, and $R(1,1,-1)$.

$\vec{PQ} = \langle -2,-2,-2\rangle \approx \langle 1,1,1\rangle$

$\vec{PR} = \langle 0,-1,-4\rangle \approx \langle 0,1,4\rangle$

$$\vec{n} = \vec{PQ}\times\vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 0 & 1 & 4 \end{vmatrix} = \langle 3,-4,1\rangle$$

$\vec{n}\cdot\vec{Q} = 3(-1)+(-4)(0)+1(1) = -3+1 = -2 = -d$

so Π: $\boxed{3x-4y+z+2=0}$

2. Find a parametrization of the curve of intersection of the cylinder $x^2+y^2=9$ and the plane $x-z=1$.

cylinder: $x = 3\cos\theta$

$y = 3\sin\theta$

plane: $z = x-1 = 3\cos\theta - 1$

so, $\boxed{\vec{r}(\theta) = \langle 3\cos\theta, 3\sin\theta, 3\cos\theta-1\rangle}$

3. Show that the curvature of a circle of radius $a > 0$ is $\kappa = \frac{1}{a}$.

$\vec{r}(t) = \langle a\cos t, a\sin t \rangle$

$\dot{\vec{r}}(t) = \langle -a\sin t, a\cos t \rangle$

$\ddot{\vec{r}}(t) = \langle -a\cos t, -a\sin t \rangle$

$\|\dot{\vec{r}}(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = a$

$\dot{\vec{r}} \times \ddot{\vec{r}} = \langle 0, 0, a^2\sin^2 t + a^2\cos^2 t \rangle = \langle 0, 0, a^2 \rangle$

$\kappa(t) = \frac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|^3} = \frac{a^2}{a^3} = \frac{1}{a}$ ∎

4. Compute the curvature of the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at the point $P(-1, 1, -1)$.

$\dot{\vec{r}} = \langle 1, 2t, 3t^2 \rangle$ $\quad \dot{\vec{r}}(-1) = \langle 1, -2, 3 \rangle$

$\ddot{\vec{r}} = \langle 0, 2, 6t \rangle$ $\quad \ddot{\vec{r}}(-1) = \langle 0, 2, -6 \rangle$

$\|\dot{\vec{r}}(-1)\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$

$\dot{\vec{r}} \times \ddot{\vec{r}} = \langle 6, 6, 2 \rangle$

$\|\dot{\vec{r}} \times \ddot{\vec{r}}\| = \sqrt{36 + 36 + 4} = \sqrt{76}$

$\kappa(-1) = \frac{\sqrt{76}}{(\sqrt{14})^3} = \frac{\sqrt{76}}{14\sqrt{14}}$ ∎

5. Let $\mathbf{r}$ be a smooth space curve such that $\ddot{\mathbf{r}}(t) \neq 0$ for all t in the domain of $\mathbf{r}$. Prove that $\dot{\mathbf{T}}(t) \perp \mathbf{T}(t)$ for all t in the domain of $\mathbf{r}$.

$$\|\vec{T}\| = 1 \Rightarrow \vec{T}\cdot\vec{T} = 1$$

$$\frac{d}{dt}\left[\vec{T}\cdot\vec{T} = 1\right] \Rightarrow \dot{\vec{T}}\cdot\vec{T} + \dot{\vec{T}}\cdot\vec{T} = 0$$

$$\Rightarrow 2\dot{\vec{T}}\cdot\vec{T} = 0$$

$$\Rightarrow \dot{\vec{T}}\cdot\vec{T} = 0$$

$$\Rightarrow \dot{\vec{T}} \perp \vec{T} \text{ for all } t \in \text{dom}(\vec{r}). \blacksquare$$

6. Compute the second directional derivative $D^2_{\mathbf{v}}f(x,y) = D_{\mathbf{v}}[D_{\mathbf{v}}f(x,y)]$ for $f(x,y) = x^3 + 5x^2y + y^3$ in the direction of the vector $\mathbf{v} = \langle 3,4\rangle$ and evaluate it at the point $(3,2)$.

$$\vec{v} = \langle 3,4\rangle \Rightarrow \vec{u} = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$$

$$\nabla f = \langle 3x^2 + 10xy,\ 5x^2 + 3y^2\rangle$$

$$\nabla f\cdot\vec{u} = \frac{3}{5}(3x^2 + 10xy) + \frac{4}{5}(5x^2 + 3y^2)$$

$$= \frac{9}{5}x^2 + 6xy + 4x^2 + \frac{12}{5}y^2$$

$$\nabla(\nabla f\cdot\vec{u}) = \left\langle \frac{18}{5}x + 6y + 8x,\ 6x + \frac{24}{5}y\right\rangle$$

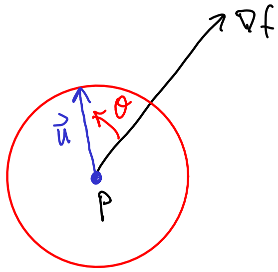
$$= \left\langle \frac{58}{5}x + 6y,\ 6x + \frac{24}{5}y\right\rangle$$

$$\nabla(\nabla f\cdot\vec{u})\cdot\vec{u} = \left\langle \frac{58}{5}x + 6y,\ 6x + \frac{24}{5}y\right\rangle\cdot\left\langle\frac{3}{5},\frac{4}{5}\right\rangle$$

$$= \frac{174}{25}x + \frac{18}{5}y + \frac{24}{5}x + \frac{96}{25}y = \left(\frac{175+120}{25}\right)x + \left(\frac{90+96}{25}\right)y$$

$$= \frac{295}{25}x + \frac{186}{25}y\Big|_{(3,2)} = \frac{885}{25} + \frac{372}{25} = \frac{1257}{25} = 50\frac{7}{25} = \boxed{50.28}$$

7. Prove the theorem: Suppose f is a differentiable function of at least 2 variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $\|\nabla f(\mathbf{x})\|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.



$$D_{\vec{u}}f(P) = \nabla f(P)\cdot\vec{u}$$
$$= \|\nabla f(P)\|\cdot\|\vec{u}\|\cos\theta$$
$$= \|\nabla f(P)\|\cos\theta$$

This is maximized when $\cos\theta = 1$, hence when $\theta = 0$.
Therefore $D_{\vec{u}}f(P)$ is maximized when $\vec{u}$ is in the same direction as $\nabla f(P)$.
The maximum value is $\max\left(D_{\vec{u}}f(P)\right) = \|\nabla f(P)\|$. ∎

8. Let $f = f(x, y)$ be a smooth function and suppose $x = r\cos\theta$, $y = r\sin\theta$. Use the chain rule to write an expression for $\dfrac{\partial^2 f}{\partial r\partial\theta}$ in terms of the x- and y- partial derivatives of f.

$$\frac{\partial f}{\partial\theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\theta}$$
$$= \frac{\partial f}{\partial x}\cdot(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)$$

$$\frac{\partial^2 f}{\partial r\,\partial\theta} = \frac{\partial}{\partial r}\left[\frac{\partial f}{\partial\theta}\right] = \frac{\partial}{\partial r}\left[-r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}\right]$$
$$= -\sin\theta\frac{\partial f}{\partial x} - r\sin\theta\left[\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial r}\right] + \cos\theta\frac{\partial f}{\partial y} + r\cos\theta\left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial r}\right)$$
$$= -\sin\theta\frac{\partial f}{\partial x} - r\sin\theta\left[\cos\theta\frac{\partial^2 f}{\partial x^2} + \sin\theta\frac{\partial^2 f}{\partial y\partial x}\right] + \cos\theta\frac{\partial f}{\partial y} + r\cos\theta\left[\cos\theta\frac{\partial^2 f}{\partial x\partial y} + \sin\theta\frac{\partial^2 f}{\partial y^2}\right]$$

$$\frac{\partial^2 f}{\partial r\,\partial\theta} = \boxed{-\sin\theta\frac{\partial f}{\partial x} + \cos\theta\frac{\partial f}{\partial y} - r\sin\theta\cos\theta\frac{\partial^2 f}{\partial x^2} + \left(r\cos^2\theta - r\sin^2\theta\right)\frac{\partial^2 f}{\partial x\partial y} + r\sin\theta\cos\theta\frac{\partial^2 f}{\partial y^2}}.$$

9. Consider the helix $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, -t\rangle$. Find the unit tangent, normal, and binormal vectors at $t = \pi$.

$\dot{\vec{r}}(t) = \langle -2\sin t, 2\cos t, -1\rangle$

$\|\dot{\vec{r}}(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}$

$\vec{T}(t) = \left\langle \frac{-2}{\sqrt{5}}\sin t, \frac{2}{\sqrt{5}}\cos t, \frac{-1}{\sqrt{5}} \right\rangle$

$\boxed{\vec{T}(\pi) = \left\langle 0, \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle}$

$\dot{\vec{T}}(t) = \left\langle \frac{-2}{\sqrt{5}}\cos t, \frac{-2}{\sqrt{5}}\sin t, 0 \right\rangle$

$\|\dot{\vec{T}}(t)\| = \frac{2}{\sqrt{5}}$

$\vec{N}(t) = \langle -\cos t, -\sin t, 0\rangle$

$\boxed{\vec{N}(\pi) = \langle 1, 0, 0\rangle}$

$\boxed{\vec{B}(\pi) = \vec{T}(\pi) \times \vec{N}(\pi) = \left\langle 0, \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle}$

10. Find the absolute maximum and absolute minimum values of the function $f(x, y) = 9 - x^2 - 2x - y^2 + 4y$ on the domain $x^2 + y^2 \leq 25$. You may leave your answers as reduced fractions.

Interior :

$\nabla f = \langle -2x-2, -2y+4 \rangle$

$\nabla f = \vec{0}$: $\begin{cases} -2x-2=0 & \Rightarrow x=-1 \\ -2y+4=0 & \Rightarrow y=2 \end{cases}$

C.P. @ $(-1,2)$

$f(-1,2) = 9-(-1)^2-2(-1)-(2)^2+4(2)$
$= 9-1+2-4+8 = 19-5 = 14$

Boundary: $\nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g$: $\left(\begin{array}{l} -2x-2 = 2\lambda x \\ -2y+4 = 2\lambda y \\ x^2+y^2=25 \end{array} \right.$

$\dfrac{-2x-2}{2x} = \dfrac{-2y+4}{2y} \Rightarrow \cancel{-4xy} - 4y = \cancel{-4xy} + 8x$

$y = -2x$

$x^2+y^2=25 \Rightarrow x^2+(-2x)^2=25$

$5x^2=25$

$x = \pm\sqrt{5} \qquad y = \mp 2\sqrt{5}$

$f(\sqrt{5}, -2\sqrt{5}) = 9-(\sqrt{5})^2-2\sqrt{5}-(-2\sqrt{5})^2+4(-2\sqrt{5})$
$= 9-5-2\sqrt{5}-20-8\sqrt{5} = -16-10\sqrt{5}$

$f(-\sqrt{5}, 2\sqrt{5}) = 9-(-\sqrt{5})^2-2(-\sqrt{5})-(2\sqrt{5})^2+4(2\sqrt{5})$
$= 9-5+2\sqrt{5}-20+8\sqrt{5} = -16+10\sqrt{5}$

$\boxed{\begin{array}{l} \text{Absolute max: } 14 \\ \text{Absolute Min: } -16-10\sqrt{5} \end{array}}$