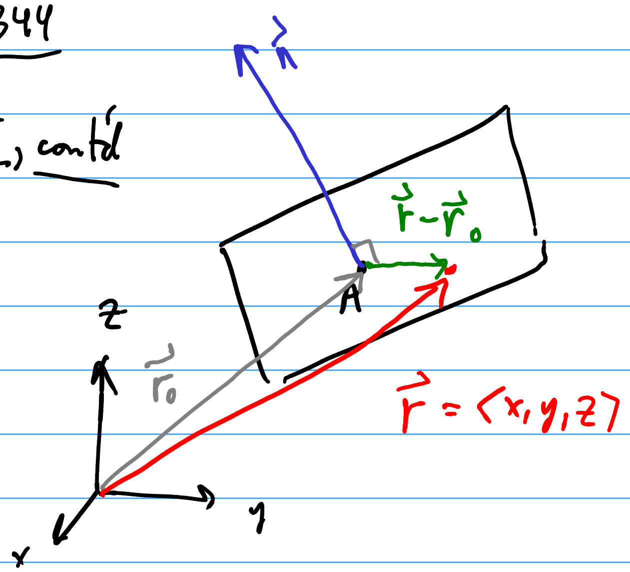
Math 344

§12.5, cont'd



$A(x_0, y_0, z_0)$

$\vec{n} = \langle a, b, c\rangle$

$\vec{r}_0 = \langle x_0, y_0, z_0\rangle$

We know $(\vec{r}-\vec{r}_0) \perp \vec{n}$, therefore $\boxed{(\vec{r}-\vec{r}_0)\cdot\vec{n} = 0.}$ $(*)$

In coordinates,

$$\langle x-x_0, y-y_0, z-z_0\rangle \cdot \langle a, b, c\rangle = 0$$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$ax + by + cz + \underbrace{(-ax_0 - by_0 - cz_0)}_{d} = 0$$

$(*)$ $\boxed{ax + by + cz + d = 0}$ where $d = -\vec{n}\cdot\vec{r}_0$

Ex. $P(1,3,2)$ $Q(3,-1,6)$ $R(5,2,0)$

$\vec{n} = \vec{PQ} \times \vec{PR}$ $\quad \vec{PQ} = \langle 2,-4,4\rangle \rightsquigarrow \langle 1,-2,2\rangle$

$PR = \langle 4,-1,-2\rangle$

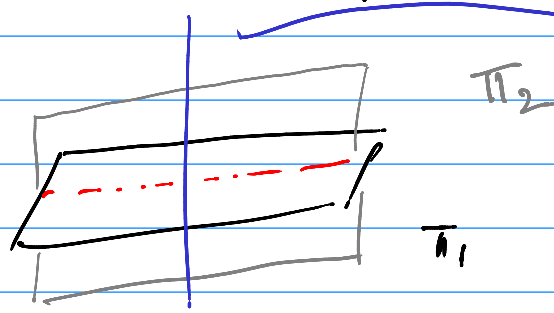
$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 2 \\ 4 & -1 & -2 \end{vmatrix} = \langle 6, 10, 7\rangle$$

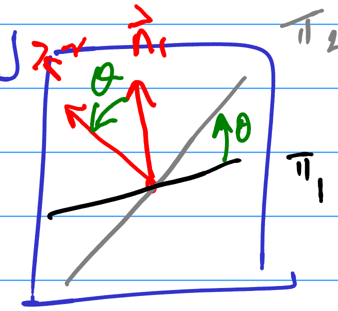
$\Pi: \quad 6x + 10y + 7z - 50 = 0 \qquad \vec{r}_0 = \langle 5,2,0 \rangle$

$$d = -\vec{n}\cdot\vec{r}_0$$
$$= -\langle 6,10,7\rangle \cdot \langle 5,2,0\rangle$$
$$= -(30+20+0) = -50$$

Two planes are parallel iff their normal vectors are parallel.

If two planes intersect, they share a common line.





Def'n. The angle between two intersecting planes is defined to be the angle between the respective normal vectors.

Ex. $\Pi_1: \; x+y+z = 1$ $\qquad$ Find the angle between these.

$\Pi_2: \; x - 2y + 3z = 1$

$\vec{n}_1 = \langle 1,1,1\rangle \qquad \vec{n}_2 = \langle 1,-2,3\rangle$

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1\cdot\vec{n}_2}{\|\vec{n}_1\|\,\|\vec{n}_2\|}\right)$$

$$\vec{n}_1\cdot\vec{n}_2 = \langle 1,1,1\rangle\cdot\langle 1,-2,3\rangle$$
$$= 1-2+3 = 2$$

$\|\vec{n}_1\| = \sqrt{3} \qquad \|\vec{n}_2\| = \sqrt{14}$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{3}\sqrt{14}}\right) \approx 72^\circ$$

Ex. $\Pi_1: \quad x+y+z=1$

$\Pi_2: \quad x-2y+3z=1$

Find an equation of the line of intersection of these.

Need: a direction vector $\vec{v} = \vec{n}_1 \times \vec{n}_2$
and a point on ℓ.

$\vec{n}_1 = \langle 1,1,1 \rangle \qquad \vec{n}_2 = \langle 1,-2,3 \rangle$

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \langle 5,-2,-3 \rangle$$

set $x=0$: $\left\{ \begin{array}{l} 0 + 2(y+z=1) \\ 0 - 2y + 3z = 1 \end{array} \right\}$ solve for y,z

$0 + 5z = 3$

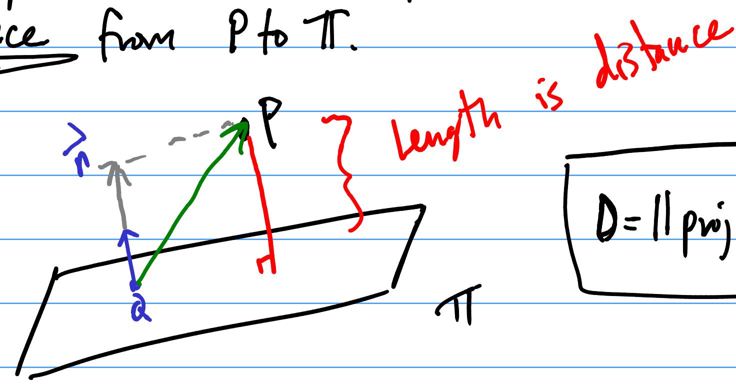
$P(0, 2/5, 3/5)$

$z = 3/5$

$y = 1 - z = 1 - 3/5 = 2/5$

The line is: $\begin{cases} x = 5t \\ y = 2/5 - 2t \\ z = 3/5 - 3t \end{cases}$ parametric eq'ns of ℓ.

Idea. Given a plane Π and a point P not on Π, compute the distance from P to Π.



$$\boxed{D = \| \mathrm{proj}_{\vec{n}} \vec{QP} \|}$$

$\vec{n} = \langle a, b, c\rangle \qquad Q(x_0, y_0, z_0) \qquad P(x_1, y_1, z_1)$

$$\vec{QP} = \langle x_1 - x_0,\, y_1 - y_0,\, z_1 - z_0\rangle$$

$$\text{proj}_{\vec n}\vec{QP} = \frac{\vec{n}\cdot\vec{QP}}{\|\vec n\|}\,\frac{\vec n}{\|\vec n\|} \quad \text{and} \quad \|\text{proj}_{\vec n}\vec{QP}\| = \left|\frac{\vec{QP}\cdot\vec n}{\|\vec n\|}\right|$$

$$\vec{QP}\cdot\vec{n} = a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0) = ax_1 + by_1 + cz_1 + d$$

$$* \quad \boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

Ex. $10x + 2y - 2z = 5 \quad \Pi_1$ These planes are parallel.
$5x + y - z = 1 \quad \Pi_2$

From one plane, pick a point: From Π_2 $(0,1,0)$ P

From Π_1, $\vec n = \langle 10, 2, -2\rangle \quad d = -5$

$$D = \frac{|\cancel{10(0)} + 2(1) - \cancel{2(0)} - 5|}{\sqrt{10^2 + 4 + 4}} = \frac{3}{\sqrt{108}} = \frac{3}{2\sqrt{27}}$$

$$= \frac{\cancel{3}}{2\cdot\cancel{3}\cdot\sqrt3} = \frac{1}{2\sqrt3}$$

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§12.6 Quadric Surfaces

$ax^2 + by^2 + cz^2 = d \leftarrow$ general formula

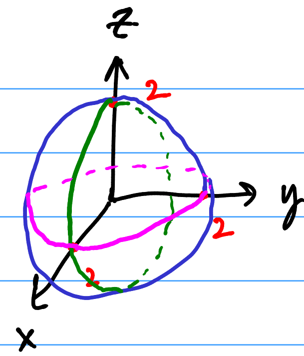
Ex. $x^2+y^2+z^2=4$

slice $x=0$: $y^2+z^2=4$

circle w/ rad. 2

$y=0$: $x^2+z^2=4$

$z=0$: $x^2+y^2=4$

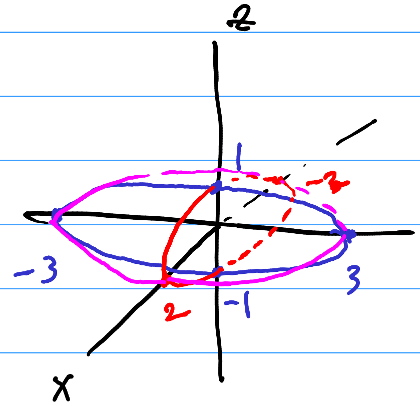


sphere w/ rad 2.

Ex. $\frac{x^2}{4}+\frac{y^2}{9}+z^2=1$

$x=0$: $\frac{y^2}{9}+z^2=1$

$y=0$: $\frac{x^2}{4}+z^2=1$



Ellipsoid

Ex. $x^2+y^2=z$

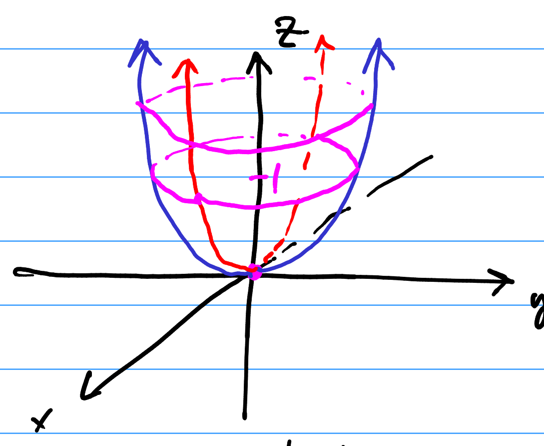
$z=0$: $x^2+y^2=0$

$z=1$: $x^2+y^2=1$

$z=4$: $x^2+y^2=4$

$x=0$: $y^2=z$

$y=0$: $x^2=z$



circular paraboloid

or elliptic

Ex. $x^2 - y^2 = z$

$x=0$: $-y^2 = z$

$y=0$: $x^2 = z$

$z=0$: $x^2 - y^2 = 0$

$x^2 = y^2 \rightarrow |x| = |y|$