M 344 8/28

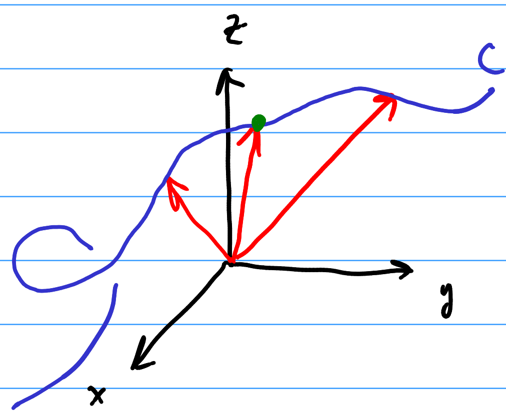
Unit Exam I (§12.5, 12.6 only) open from 8/28 @ 12:15 pm
closes at 8/31 @ 11:59 pm

Ch 13 – Vector Functions

Def'n. A vector function is a function that takes in a real number as its argument and outputs a vector.

$$f: \mathbb{R} \longrightarrow \mathbb{R}^n$$

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$



The output vectors can be thought of as position vectors for their terminal pts. If $\vec{r}$ is nice, then these vectors trace out a space curve.

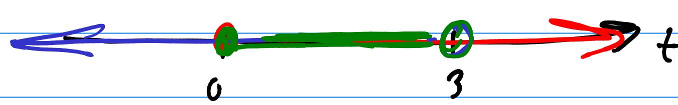
Ex. $\vec{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$

$\text{dom}(\vec{r}) = \text{dom}(x) \cap \text{dom}(y) \cap \text{dom}(z)$ "overlaps"

x: $\text{dom}(x) = \mathbb{R}$

y: $3-t>0 \Rightarrow t<3$

z: $t \geq 0$



$\text{dom}(\vec{r}) = [0, 3)$

Limits: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$$\lim_{t\to a} \vec{r}(t) = \lim_{t\to a}\left(x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}\right)$$

$$= \lim_{t\to a}\left(x(t)\vec{i}\right) + \lim_{t\to a}\left(y(t)\vec{j}\right) + \lim_{t\to a}\left(z(t)\vec{k}\right)$$

$$= \left(\lim_{t\to a} x(t)\right)\vec{i} + \left(\lim_{t\to a} y(t)\right)\vec{j} + \left(\lim_{t\to a} z(t)\right)\vec{k}$$

So,

$$\boxed{\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \right\rangle}$$

(*) In order for $\lim_{t\to a} \vec{r}(t)$ to exist, each component limit must exist.

Ex. $\vec{r}(t) = \left\langle 1+t^3, te^{-t}, \frac{\sin t}{t} \right\rangle$

$$\lim_{t\to 0} \vec{r}(t) = \left\langle \lim_{t\to 0}(1+t^3), \lim_{t\to 0} te^{-t}, \lim_{t\to 0} \frac{\sin t}{t} \right\rangle$$

x $\lim_{t\to 0} 1+t^3 = 1+0^3 = 1$

y $\lim_{t\to 0} te^{-t} = 0\cdot e^{-0} = 0\cdot 1 = 0$

z $\lim_{t\to 0} \frac{\sin t}{t} = \frac{\sin 0}{0} = \frac{0}{0} \overset{L'H}{=} \lim_{t\to 0} \frac{\cos t}{1} = \lim_{t\to 0} \cos t = \cos 0 = 1$

$$\lim_{t\to 0} \vec{r}(t) = \langle 1, 0, 1 \rangle$$

Defn. (Calc I) A function f is continuous at $x=a$ if and only if

$$\lim_{x\to a} f(x) = f(a)$$

Defn. (Calc III)

A vector function $\vec{r}$ is <u>continuous</u> at $t=a$ iff

$$\lim_{t\to a} \vec{r}(t) = \vec{r}(a).$$

Ex. Describe the space curve traced out by the vector function

$$\vec{r}(t) = \langle 1+t,\ 2+5t,\ -1+6t \rangle$$

$$\begin{cases} x = 1+t \\ y = 2+5t \\ z = -1+6t \end{cases}$$

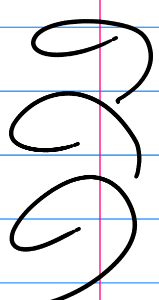
<u>Line!</u>

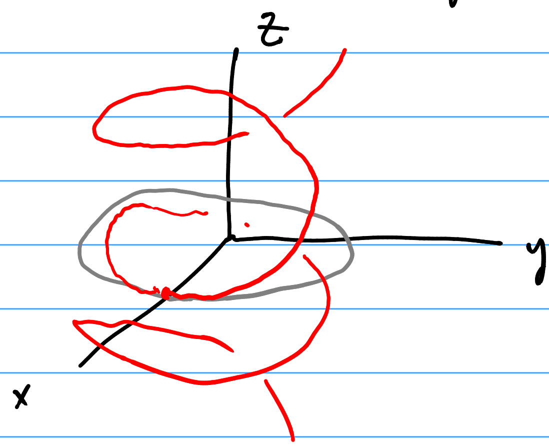
$P(1,2,-1)$

$\vec{v} = \langle 1,5,6 \rangle$

Ex. $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k} = \langle \underbrace{\cos t, \sin t}, t \rangle$

<u>Helix</u>





$x = \cos t$

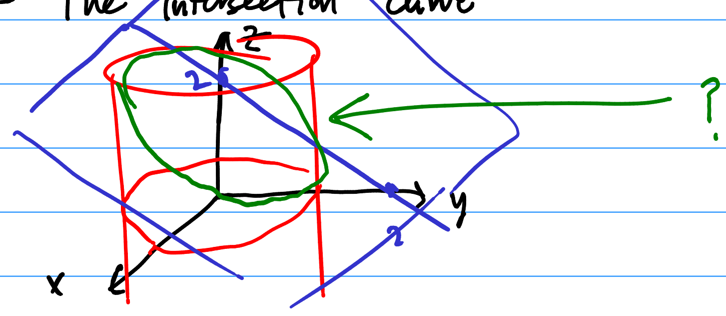
$y = \sin t$ $\Rightarrow \langle \cos t, \sin t \rangle$

"I magine"

Ex. Consider the cylinder $x^2+y^2=1$ and the plane $y+z=2$.

Find an equation of the intersection curve

$z = -y+2$



The cylinder can be parametrized by

$$\left\{\begin{array}{l} x=\cos t \\ y=\sin t \\ z=2-\sin t \end{array}\right.$$

$$\vec{r}(t)=\langle \cos t, \sin t, z(t)\rangle$$

Now restrict z to ensure we're on the ellipse.

$$z=-y+2=-\sin t+2$$

$$\vec{r}(t)=\langle \cos t, \sin t, 2-\sin t\rangle$$

More Examples:

1. Toroidal Spiral: $\left\{\begin{array}{l} x=(4+\sin(20t))\cos t \\ y=(4+\sin(20t))\sin t \\ z=\cos(20t) \end{array}\right.$

2. Trefoil knot: $\left\{\begin{array}{l} x=(2+\cos(\frac{3}{2}t))\cos t \\ y=(2+\cos(\frac{3}{2}t))\sin t \\ z=\sin(\frac{3}{2}t) \end{array}\right.$

3. Twisted Cubic: $\left\{\begin{array}{l} x=t \\ y=t^2 \\ z=t^3 \end{array}\right.$

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§13.2 – Derivatives and Integrals.

Def'n. (Calc I) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Def'n (Calc III) Let $\vec{r}$ be a vector function. The derivative

$$\dot{\vec{r}}(t)=\vec{r}\,'(t)=\lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$

$$\dot{\vec{r}}(t) = \left\langle \lim_{h\to 0} \frac{x(t+h)-x(t)}{h}, \lim_{h\to 0} \frac{y(t+h)-y(t)}{h}, \lim_{h\to 0} \frac{z(t+h)-z(t)}{h} \right\rangle$$

$$\dot{\vec{r}}(t) = \langle \dot{x}(t), \dot{y}(t), \dot{z}(t) \rangle$$

recall: $\left\{ \begin{array}{l} \dot{\vec{r}} = \frac{d\vec{r}}{dt} \\ \dot{x} = \frac{dx}{dt} \end{array} \right\}$

Ex. $\vec{r}(t) = \langle 1+t^3, te^{-t}, \sin 2t \rangle$

$$\dot{\vec{r}}(t) = \langle 3t^2, e^{-t} - te^{-t}, 2\cos(2t) \rangle$$

Q. In calc I, the derivative is the slope of the tan line to the curve at the pt.
What about in 3D?

$\underbrace{x^2+4x}_{} +$