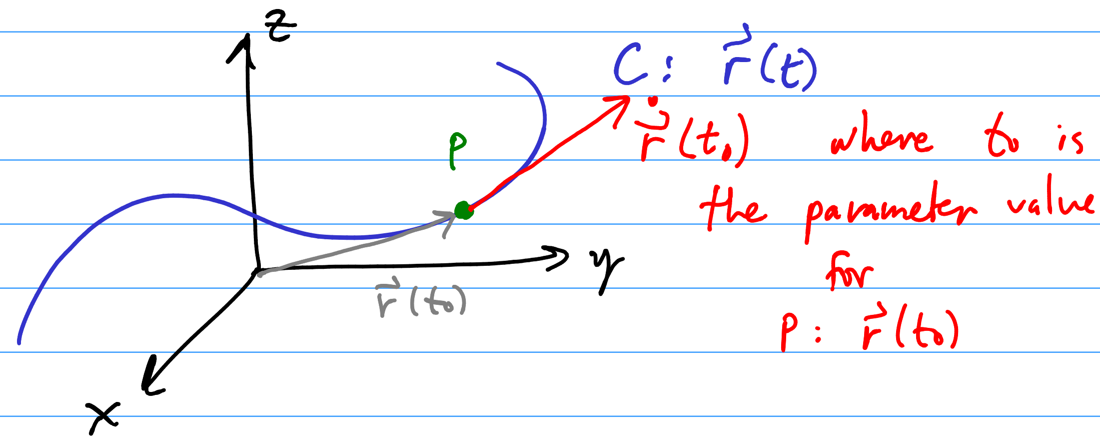
M344 8/30

§13.2 – Derivatives and integrals of vector functions



Def'n. The tangent line to the space curve C parametrized by a vector function $\vec{r}$ at the point $P = \vec{r}(t_0)$ is

$$\vec{u}(t) = \vec{r}(t_0) + t\,\dot{\vec{r}}(t_0) \qquad (*)$$

Ex. $\begin{cases} x = 2\cos t \\ y = \sin t \\ z = t \end{cases}$ at $\boxed{P(0, 1, \pi/2)}$

$$0 = 2\cos t$$
$$1 = \sin t$$
$$\pi/2 = t \quad \leftarrow \quad t_0$$

$\dot{\vec{r}} = \langle \dot{x}, \dot{y}, \dot{z} \rangle$ $\qquad \begin{cases} \dot{x} = -2\sin t \\ \dot{y} = \cos t \\ \dot{z} = 1 \end{cases}$ $\qquad \boxed{\dot{\vec{r}}(\pi/2) = \langle -2, 0, 1 \rangle}$

The tan. line is given by:

$$\boxed{\vec{u}(t) = \langle -2t, 1, \pi/2 + t \rangle}$$

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt}, \qquad \ddot{\vec{r}} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2} = (\dot{\vec{r}})^{\cdot}$$

Differentiation Rules

Thm. $\vec{u}, \vec{v}$ be vector functions, c a scalar, f a real-valued function

1. $\frac{d}{dt}\left[\vec{u}(t) \pm \vec{v}(t)\right] = \dot{\vec{u}}(t) \pm \dot{\vec{v}}(t)$
2. $\frac{d}{dt}\left[c\,\vec{u}(t)\right] = c\,\dot{\vec{u}}(t)$

$f(t)\langle x(t), y(t), z(t)\rangle = \langle f(t)x(t), f(t)y(t), \ldots\rangle$

3. $\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = \dot{f}(t)\vec{u}(t) + f(t)\dot{\vec{u}}(t)$
4. $\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \dot{\vec{u}}(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\dot{\vec{v}}(t)$
5. $\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \dot{\vec{u}}(t)\times\vec{v}(t) + \vec{u}(t)\times\dot{\vec{v}}(t)$ order matters!
6. $\frac{d}{dt}\left[\vec{u}(f(t))\right] = \dot{\vec{u}}(f(t))\,\dot{f}(t)$ Chain Rule.

RE. Prove some of these for vector functions in $\mathbb{R}^3$.

Thm. If $\|\vec{r}(t)\| = C$ where C is a constant, then $\vec{r}(t) \perp \dot{\vec{r}}(t)$.

Proof. $\|\vec{r}(t)\|^2 = C^2$

$C = \|\vec{r}(t)\|^2 = \vec{r}(t)\cdot\vec{r}(t)$

$\frac{d}{dt}\left(C = \vec{r}\cdot\vec{r}\right)$

$0 = \frac{d}{dt}(\vec{r}\cdot\vec{r}) = \dot{\vec{r}}\cdot\vec{r} + \vec{r}\cdot\dot{\vec{r}} = 2(\dot{\vec{r}}\cdot\vec{r})$

$\Rightarrow \dot{\vec{r}}\cdot\vec{r} = 0$ for all t. So $\vec{r} \perp \dot{\vec{r}}$ ∎

Sphere:

Integrals!

Calc I. f is continuous $[a,b]$. Then, the area under f is given

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{n=0}^{N} f(x_n^*)\,\Delta x_n \quad \longleftarrow$$

Calc III. For a vector function $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, the integral $\vec{r}$ is

$$(*) \quad \int_{t=a}^{b} \vec{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

or

$$\int \vec{r}(t)\,dt = \left\langle \int x\,dt, \int y\,dt, \int z\,dt \right\rangle + \vec{c}$$

Fundamental Theorem of Calculus:

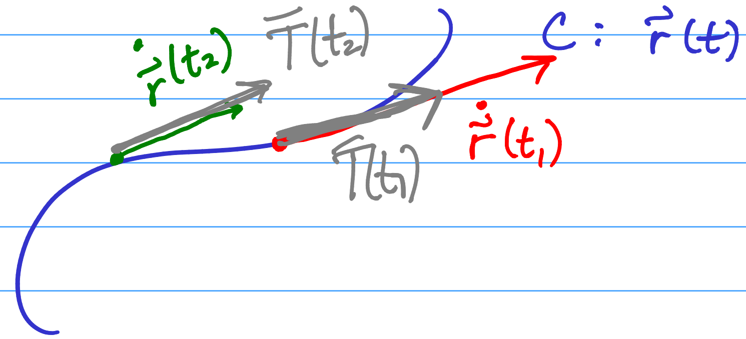
If $\vec{R}(t)$ is a vector function s.t. $\dot{\vec{R}}(t) = \vec{r}(t)$, then

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{c}$$

Ex. $\vec{r} = \langle 2\cos t, \sin t, 2t\rangle$

$$\int \vec{r}\,dt = \langle 2\sin t + c_1, -\cos t + c_2, t^2 + c_3\rangle = \langle 2\sin t, -\cos t, t^2\rangle + \vec{C}$$

§13.3 Arc length and Curvature

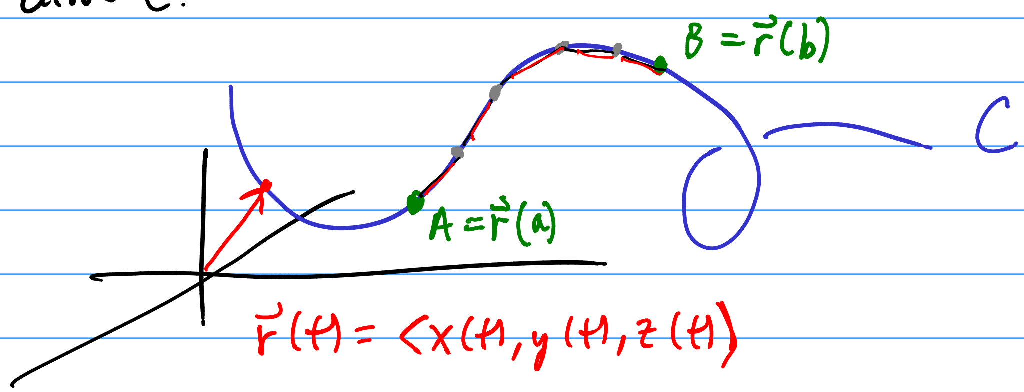


Def'n. A vector function $\vec r$ is said to be smooth if $\dot{\vec r}$ exists at each point, and $\dot{\vec r}(t) \neq \vec 0$ for any t.

Def'n. The (unit) tangent vector to a smooth space curve C at a point t_0 is given by

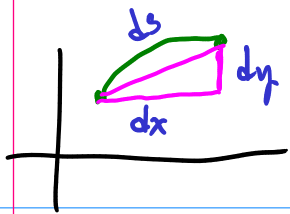
$$\boxed{\vec T(t_0) = \frac{\dot{\vec r}(t_0)}{\|\dot{\vec r}(t_0)\|}} \qquad (*)$$

Let $\vec r$ be a smooth vector function parametrizing a space curve C.



The arc length of the curve between A and B is given by

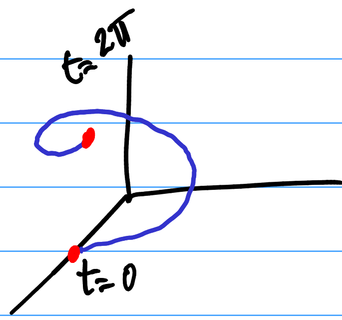
$$s = \int_a^b \|\dot{\vec r}(t)\|\,dt = \int_a^b \|\langle \dot x(t), \dot y(t), \dot z(t) \rangle\|\,dt$$



$$s = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}dt\right)^2 + \left(\frac{dy}{dt}dt\right)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex. $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ Find the length from the $(1,0,0)$ to the pt $P(1,0,2\pi)$



$$s = \int \|\dot{\vec{r}}\|\, dt$$

$$\dot{\vec{r}} = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\dot{\vec{r}}\| = \sqrt{\sin^2 t + \cos^2 t + 1^2} = \sqrt{2}$$

$$s = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}\, t \Big|_0^{2\pi} = \sqrt{2}(2\pi - 0) = \boxed{2\pi\sqrt{2} = s}$$

The arc length function starting at $t = t_0$ is given by

$$s(t) = \int_{t_0}^{t} \|\dot{\vec{r}}(u)\|\, du$$

Ex. for the helix

$$s(t) = \int_0^t \sqrt{2}\, du = \sqrt{2}u \Big|_0^t = \sqrt{2}\, t$$

we get $s = \sqrt{2}\, t \rightarrow$ solve for $t = \frac{s}{\sqrt{2}}$ and plug in to reparametrize:

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$$\boxed{\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle} \quad (*)$$

"parametrized by arc length"

Ex. $s = \displaystyle\int_{t_0}^{t} \|\dot{\vec{r}}(u)\|\, du$

$$\frac{ds}{dt} = \frac{d}{dt}\int_{t_0}^{t} \|\dot{\vec{r}}(u)\|\, du = \boxed{\|\dot{\vec{r}}(t)\| = \frac{ds}{dt}}$$