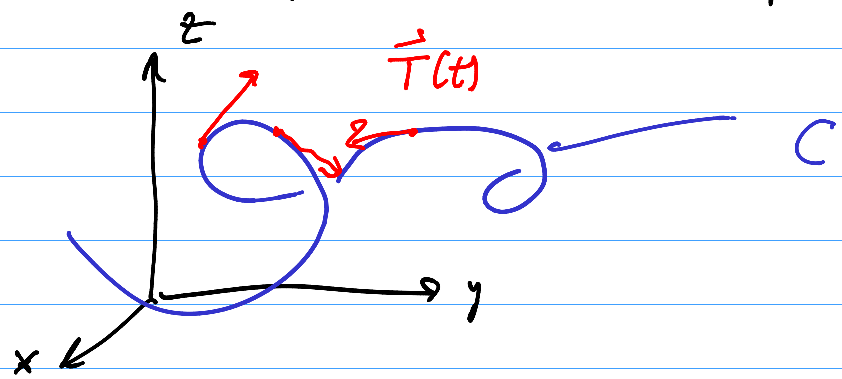
M344 9/4

§13.3 - Arc length and Curvature

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ parametrization of a space curve C.

"smooth": $\dot{\vec{r}}(t) \neq \vec{0}$

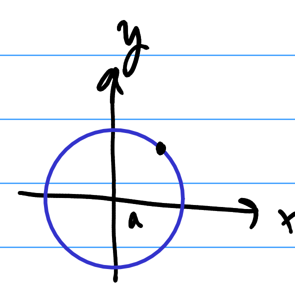


$$\boxed{\vec{T} = \frac{\dot{\vec{r}}(t)}{\|\dot{\vec{r}}(t)\|}} \Rightarrow \|\vec{T}\| = 1$$

Defn. The curvature of a smooth space C is the function defined by

$$\boxed{\kappa(s) = \left\| \frac{d\vec{T}}{ds} \right\|} \quad (*)$$

Ex. circle of radius $a > 0$.
Find the curvature.



$\vec{r}(t) = \langle a\cos t, a\sin t \rangle$

$$s(t) = \int_{t_0}^{t} \|\dot{\vec{r}}(u)\|\, du$$

$t_0 = 0$

$\dot{\vec{r}}(t) = \langle -a\sin t, a\cos t \rangle$

$\|\dot{\vec{r}}(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t}$
$= \sqrt{a^2}\sqrt{\sin^2 t + \cos^2 t}$

$$= |a|$$
$$= a$$

$$s(t) = \int_0^t a \, du = at \qquad s = at \Rightarrow t = \frac{s}{a}$$

$$\vec{r}(t) = \langle a\cos t, \ a\sin t\rangle \rightarrow \vec{r}(s) = \left\langle a\cos\left(\frac{s}{a}\right), \ a\sin\left(\frac{s}{a}\right)\right\rangle$$

$$\vec{T} = \frac{\dot{\vec{r}}}{\|\dot{\vec{r}}\|} = \frac{\frac{d\vec{r}}{ds}}{\|d\vec{r}/ds\|} \qquad \frac{d\vec{r}}{ds} = \left\langle -\frac{a}{a}\sin\left(\frac{s}{a}\right), \ \frac{a}{a}\cos\left(\frac{s}{a}\right)\right\rangle$$

$$= \left\langle -\sin\left(\frac{s}{a}\right), \ \cos\left(\frac{s}{a}\right)\right\rangle$$

$$\left\|\frac{d\vec{r}}{ds}\right\| = \sqrt{\sin^2\left(\frac{s}{a}\right) + \cos^2\left(\frac{s}{a}\right)} = 1$$

so, $\vec{T}(s) = \frac{d\vec{r}}{ds} = \left\langle -\sin\left(\frac{s}{a}\right), \ \cos\left(\frac{s}{a}\right)\right\rangle$

$$\frac{d\vec{T}}{ds} = \left\langle -\frac{1}{a}\cos\left(\frac{s}{a}\right), \ -\frac{1}{a}\sin\left(\frac{s}{a}\right)\right\rangle$$

$$\kappa(s) = \left\|\frac{d\vec{T}}{ds}\right\| = \sqrt{\frac{1}{a^2}\cos^2\left(\frac{s}{a}\right) + \frac{1}{a^2}\sin^2\left(\frac{s}{a}\right)} = \sqrt{\frac{1}{a^2}}\sqrt{\cos^2\left(\frac{s}{a}\right) + \sin^2\left(\frac{s}{a}\right)}$$

$$\boxed{\kappa(s) = \frac{1}{a}} \quad *$$

Summary, The curvature of a circle w/ radius $a>0$ is constant, and $\kappa(s) = 1/a$.

Recall: $\|\dot{\vec{r}}(t)\| = \frac{d}{dt}\int_{t_0}^{t} \|\dot{\vec{r}}(u)\| \, du = \frac{ds}{dt}$

$$\kappa(s) = \left\|\frac{d\vec{T}}{ds}\right\| = \left\|\frac{d\vec{T}/dt}{ds/dt}\right\| = \left\|\frac{\dot{\vec{T}}}{\|\dot{\vec{r}}\|}\right\| = \frac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|}$$

Thm. $$\kappa(t) = \frac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|}$$ ∎

Ex. $\vec{r}(t) = \langle a\cos t, a\sin t\rangle$

$\dot{\vec{r}}(t) = \langle -a\sin t, a\cos t\rangle$ $\quad \|\dot{\vec{r}}(t)\| = a$

$$\vec{T}(t) = \frac{\dot{\vec{r}}}{\|\dot{\vec{r}}\|} = \left\langle \frac{-a\sin t}{a}, \frac{a\cos t}{a}\right\rangle = \langle -\sin t, \cos t\rangle$$

$$\dot{\vec{T}} = \langle -\cos t, -\sin t\rangle, \quad \|\dot{\vec{T}}\| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$\kappa(t) = \frac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|} = \frac{1}{a}$$

Thm. $$\kappa(t) = \frac{\|\dot{\vec{r}}(t) \times \ddot{\vec{r}}(t)\|}{\|\dot{\vec{r}}(t)\|^3}$$

Proof: Work out the RHS. ∎ (RE)

Ex. $\vec{r}(t) = \langle a\cos t, a\sin t, 0\rangle$ Do it!

Ex. Twisted cubic: $\vec{r}(t) = \langle t, t^2, t^3\rangle$

Find $\kappa(0)$, $\quad \vec{r}(0) = \langle 0, 0, 0\rangle$

$\dot{\vec{r}}(t) = \langle 1, 2t, 3t^2\rangle$ $\quad \|\dot{\vec{r}}\| = \sqrt{1+4t^2+9t^4}$

$\ddot{\vec{r}}(t) = \langle 0, 2, 6t\rangle$

$\dot{\vec{r}}(0) = \langle 1, 0, 0\rangle$ $\quad \|\dot{\vec{r}}(0)\| = \|\langle 1,0,0\rangle\| = 1$

$$\ddot{\vec{r}}(0) = \langle 0, 2, 0 \rangle$$

$$\dot{\vec{r}}(0) \times \ddot{\vec{r}}(0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix} = \langle 0, 0, 2 \rangle$$

$$\kappa(0) = \frac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|^3} = \frac{\|\langle 0,0,2\rangle\|}{1^3} = \frac{2}{1} = 2.$$

Special Cases ($\mathbb{R}^2$): $\quad y = f(x)$

$$\kappa(x) = \frac{\|\dot{\vec{r}}(x) \times \ddot{\vec{r}}(x)\|}{\|\dot{\vec{r}}(x)\|^3}$$

$$\vec{r}(x) = \langle x, f(x), 0 \rangle$$

$$\dot{\vec{r}}(x) = \langle 1, f'(x), 0 \rangle \qquad \|\dot{\vec{r}}(x)\| = \sqrt{1 + (f'(x))^2}$$

$$\ddot{\vec{r}}(x) = \langle 0, f''(x), 0 \rangle \qquad \dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = \langle 0, 0, f''(x) \rangle$$

$$\|\dot{\vec{r}}(x) \times \ddot{\vec{r}}(x)\| = |f''(x)|$$

$$\boxed{\kappa(x) = \frac{|f''(x)|}{\left(\sqrt{1 + (f'(x))^2}\right)^3}}$$

Ex. $y = 4x^2$ $\qquad$ Find $\kappa(x)$

$$f(x) = 4x^2$$
$$f'(x) = 8x$$
$$f''(x) = 8$$

$$\kappa(x) = \frac{8}{\left(\sqrt{1 + 64x^2}\right)^3}$$

$\underline{Ex}$. $y = \ln(x)$

$f(x) = \ln(x)$

$f'(x) = 1/x \quad \leftarrow \quad 1 + f'(x)^2 = 1 + 1/x^2 = \frac{x^2+1}{x^2}$

$f''(x) = -1/x^2$

$\sqrt{1+f'(x)^2} = \sqrt{\frac{x^2+1}{x^2}} = \frac{\sqrt{x^2+1}}{x}$

$\kappa(x) = \frac{|f''(x)|}{\left(\sqrt{1+f'(x)^2}\right)^3} = \frac{1/x^2}{\frac{\sqrt{x^2+1}^3}{x^3}} = \frac{x}{\left(\sqrt{x^2+1}\right)^3} \leftarrow$

$\kappa'(x) = ?$

$\underline{Ex}$. $\vec{r}(t) = \langle x(t), y(t) \rangle$ parametrized plane curve.

$\kappa(t) = \frac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|^3}$

$\vec{r}(t) = \langle x(t), y(t), 0 \rangle$

$\dot{\vec{r}}(t) = \langle \dot{x}(t), \dot{y}(t), 0 \rangle \qquad \|\dot{\vec{r}}\| = \sqrt{\dot{x}^2 + \dot{y}^2}$

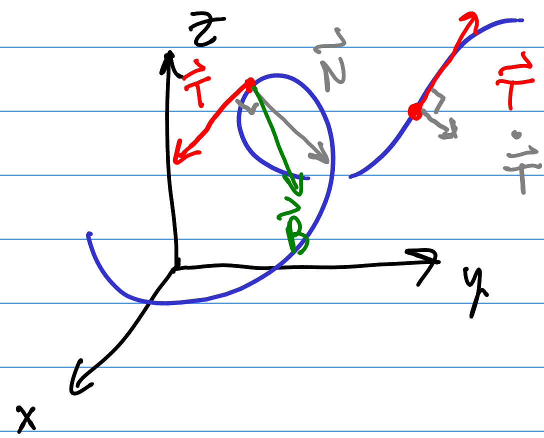
$\ddot{\vec{r}}(t) = \langle \ddot{x}(t), \ddot{y}(t), 0 \rangle$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dot{x} & \dot{y} & 0 \\ \ddot{x} & \ddot{y} & 0 \end{vmatrix}$$

$$= \langle 0, 0, \dot{x}\ddot{y} - \ddot{x}\dot{y} \rangle$$

$$\boxed{\kappa(t) = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{\left(\sqrt{\dot{x}^2 + \dot{y}^2}\right)^3}} \quad (*)$$

Frenet Frames



$C: \vec{r}(t)$

<u>Want</u>: a vector in the direction of the <u>curvature</u>.

$\vec{T} = \dfrac{\dot{\vec{r}}}{\|\dot{\vec{r}}\|}$, $\|\vec{T}\| = 1$

$\frac{d}{dt}\vec{T} \; ? \; \vec{T}$

$$\|\vec{T}\|^2 = 1^2$$

$$\frac{d}{dt}\left(\vec{T}\cdot\vec{T} = 1\right)$$

$$\dot{\vec{T}}\cdot\vec{T} + \vec{T}\cdot\dot{\vec{T}} = 0$$

$$2(\underbrace{\dot{\vec{T}}\cdot\vec{T}}_{=0}) = 0$$

Therefore, $\dot{\vec{T}} \perp \vec{T}$ for all values of t.

As long as $\dot{\vec{T}} \neq \vec{0}$, then we can make a unit vector in its direction:

$$\vec{N} = \frac{\dot{\vec{T}}}{\|\dot{\vec{T}}\|}$$ <u>Unit Normal vector</u>

$$\vec{B} = \vec{T} \times \vec{N}$$ <u>Unit Binormal vector</u>

$\underline{\text{Thm}}$. $\|\vec{B}\| = 1$

$$\|\vec{B}\| = \|\vec{T} \times \vec{N}\| = \|\vec{T}\| \, \|\vec{N}\| \sin\theta = 1 \cdot 1 \cdot \sin(\pi/2) = 1 \cdot 1 \cdot 1 = 1.$$