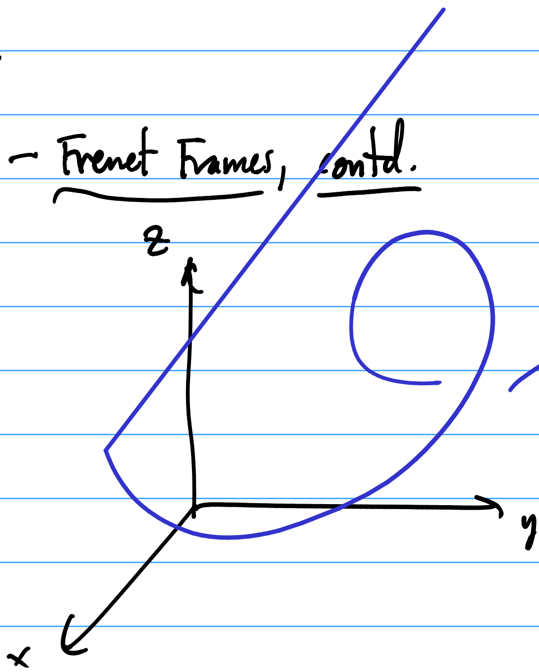
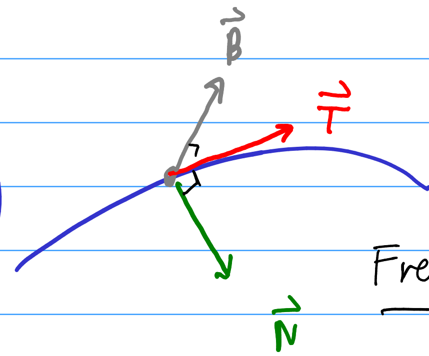
M344 9/6/18

§13.3 – Frenet Frames, contd.





$C : \vec{r}(t)$

Frenet Frame

$$\kappa(t) = \frac{\|\dot{\vec{T}}(t)\|}{\|\dot{\vec{r}}(t)\|}$$

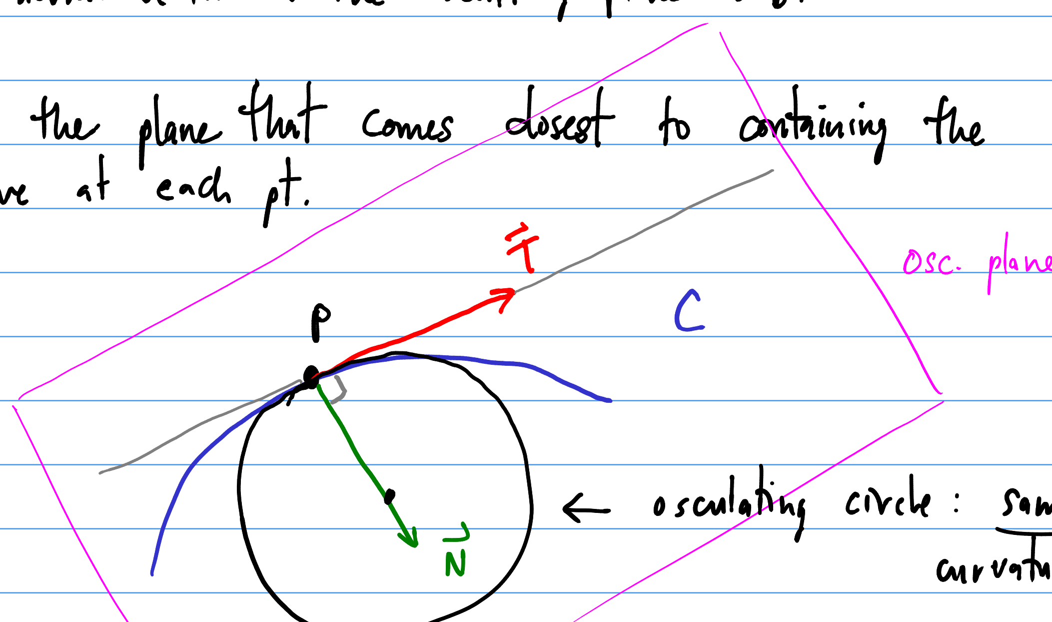
Def'n. The normal plane to the curve at any point is the plane spanned by $\vec{B}$ and $\vec{N}$.
$\vec{T}$ is the normal vector to the normal plane.
The curve is orthogonal to its normal planes at each point.

Def'n. The osculating plane to the curve at any point is the plane spanned by $\vec{T}$ and $\vec{N}$.

The normal vector to the osculating plane is $\vec{B}$.

This the plane that comes closest to containing the curve at each pt.

"Picture":

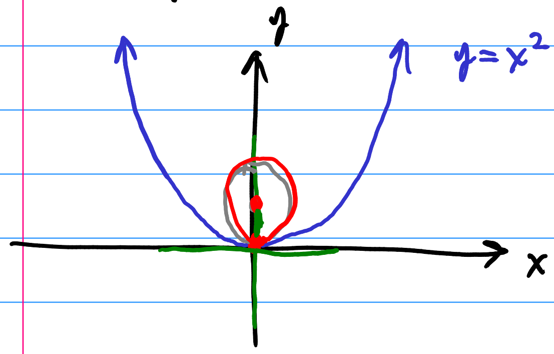


← osculating circle: same curvature.

Criteria for osculating circle:

- shares the same $\vec{T}$ and $\vec{N}$ as the curve at P.
- is in the osculating plane.
- has center on the ray determined by $\vec{N}$
- has the same curvature, $\kappa(P)$
  - the radius must then be $r = 1/\kappa$

Ex. $y = x^2$ Find the osculating circle at $(0,0)$.



$$(x-h)^2 + (y-k)^2 = r^2$$

$h = 0$

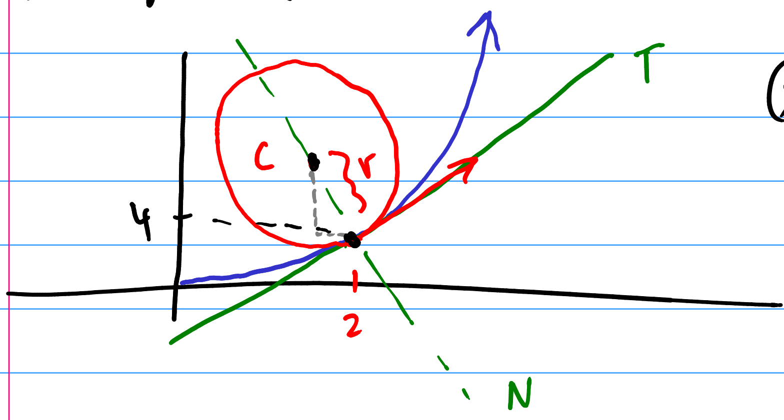
$$\boxed{x^2 + (y - 1/2)^2 = 1/4}$$

osculating circle

$$\kappa(x) = \frac{|f''(x)|}{\sqrt{1+f'(x)^2}^{\,3}} \qquad r = 1/\kappa = 1/2$$

$$\left.\begin{aligned} f(x) &= x^2 \\ f'(x) &= 2x \\ f''(x) &= 2 \end{aligned}\right\} \quad \kappa(x) = \frac{|2|}{\sqrt{1+4x^2}^{\,3}}, \quad \kappa(0) = \frac{2}{\sqrt{1+0}^{\,3}} = 2$$

Ex. $y = x^2$, @ $x = 2$



$$(x-h)^2 + (y-k)^2 = r^2$$

$$\left.\begin{aligned} f(x) &= x^2 \\ f'(x) &= 2x \\ f''(x) &= 2 \end{aligned}\right\} \quad \kappa(x) = \frac{2}{\sqrt{1+4x^2}^{\,3}}$$

$$\kappa(2) = \frac{2}{\sqrt{17}^{\,3}}, \quad r = \frac{\sqrt{17}^{\,3}}{2}$$

$x = 2$
$y = 4$

$$(2-h)^2 + (4-k)^2 = \frac{17^3}{4}$$

N: $y = f(a) + m(x-a)$

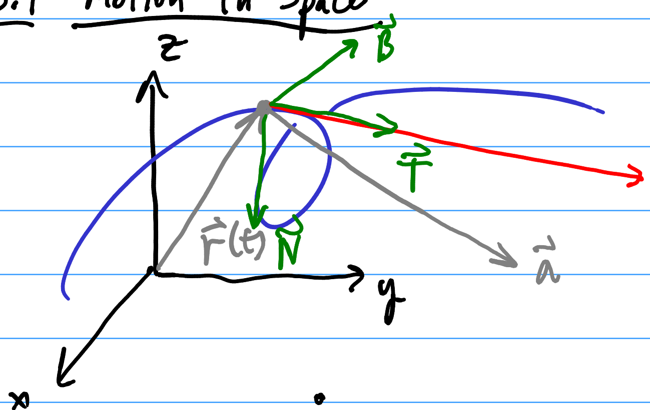
$$y = 4 - \frac{1}{f'(2)}(x-2) \qquad f'(2) = 2(2) = 4$$

$$y = 4 - 1/4(x-2)$$

$$\begin{cases} k = 4 - 1/4(h-2) \\ (2-h)^2 + (4-k)^2 = \frac{17^3}{4} \end{cases}$$ solve for h, k.

? RE. Finish it.
→ RE. Vectorize it.

13.4 Motion in Space



C: $\vec{r}(t)$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$
position of a particle at t.

$\vec{v}(t) = \dot{\vec{r}}(t) = \langle \dot{x}(t), \dot{y}(t), \dot{z}(t) \rangle$ is the velocity of the particle at "time" t.

speed: $v(t) = \|\vec{v}(t)\| = \|\dot{\vec{r}}(t)\| = \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}$

$\vec{a}(t) = \dot{\vec{v}}(t) = \ddot{\vec{r}}(t) = \langle \ddot{x}(t), \ddot{y}(t), \ddot{z}(t) \rangle$ acceleration

The acceleration at any point is in the osculating plane.

Thus, $\vec{a}(t) = \alpha \vec{T}(t) + \beta \vec{N}(t)$

$$\begin{cases} \alpha = a_T = \text{tangential acceleration} \\ \beta = a_N = \text{normal acceleration} \end{cases}$$

First, a fact. $\vec{v}(t) = \underline{\quad\quad} \vec{T}(t) = \boxed{v(t)\,\vec{T}(t) = \vec{v}(t)}$

or $\vec{v} = v\vec{T}$

$$\vec{a} = \dot{\vec{v}} = (v\vec{T})^{\boldsymbol{\cdot}} = \underline{\dot{v}}\,\vec{T} + v\,\underline{\dot{\vec{T}}}$$

$a_T = \alpha = \dot{v}(t)$

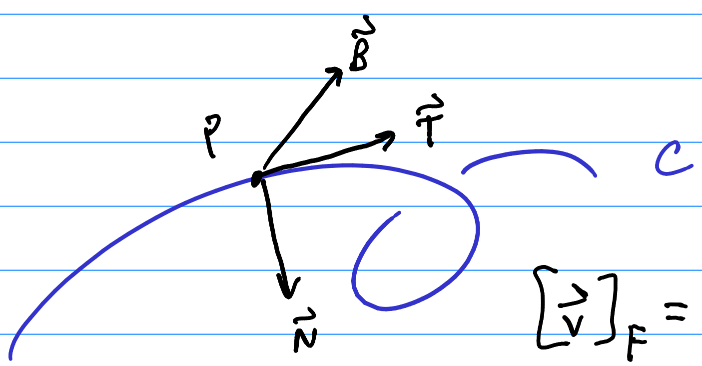
$$\dot{\vec{T}} = \underline{\|\dot{\vec{T}}\|}\,\vec{N} = \varkappa v\vec{N} \qquad \vec{N} = \frac{\dot{\vec{T}}}{\|\dot{\vec{T}}\|}$$

$$\varkappa = \left\|\frac{d\vec{T}}{ds}\right\| = \frac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|} = \varkappa \;, \quad \text{so} \quad \|\dot{\vec{T}}\| = \varkappa\|\dot{\vec{r}}\| = \varkappa\|\vec{v}\| = \varkappa v$$

Putting it all together,

$$(*) \quad \boxed{\vec{a} = \dot{v}\vec{T} + \varkappa v^2 \vec{N}}$$

$$\begin{cases} a_T = \alpha = \dot{v} \\ a_N = \beta = \varkappa v^2 \end{cases}$$



$$\left.\begin{aligned} [\vec{v}]_F &= [v, 0, 0] \\ [\vec{a}]_F &= [\dot{v}, \varkappa v^2, 0] \end{aligned}\right\}$$

RE. $\left\{ \begin{array}{l} a_T = \dot{v} = \dfrac{\dot{\vec{r}} \cdot \ddot{\vec{r}}}{\|\dot{\vec{r}}\|} \qquad \dfrac{d}{dt}\left[\|\dot{\vec{r}}\|\right] \\[2ex] a_N = \kappa v^2 = \dfrac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|} \end{array} \right.$

Ex. $\vec{r}(t) = \langle t^2, t^2, t^3 \rangle$ Find $\vec{a}(t) = \underline{a_T}\,\vec{T} + \underline{a_N}\,\vec{N}$

$$a_T = \frac{\dot{\vec{r}} \cdot \ddot{\vec{r}}}{\|\dot{\vec{r}}\|} \qquad \dot{\vec{r}} = \langle 2t, 2t, 3t^2 \rangle \qquad \dot{\vec{r}} \cdot \ddot{\vec{r}} = 4t + 4t + 18t^3$$

$$\ddot{\vec{r}} = \langle 2, 2, 6t \rangle \qquad = 8t + 18t^3$$

$$\|\dot{\vec{r}}\| = \sqrt{4t^2 + 4t^2 + 9t^4} = \sqrt{8t^2 + 9t^4}$$

$$\boxed{a_T = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}}}$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} = \langle 6t^2, -6t^2, 0 \rangle$$

$$\|\dot{\vec{r}} \times \ddot{\vec{r}}\| = \sqrt{2(6t^2)^2} = 6\sqrt{2}\, t^2$$

$$\boxed{a_N = \frac{6\sqrt{2}\, t^2}{\sqrt{8t^2 + 9t^4}}}$$