M344 9/13/18

Ex. $\lim_{\vec{x}\to\vec{0}} \frac{3x^2y}{x^2+y^2} = 0$

$\vec{x} = \langle x, y\rangle$

if $0 < \|\vec{x} - \vec{0}\| < \delta$, then $\underbrace{|f(\vec{x}) - L| < \varepsilon}$

$\|\vec{x}\| = \sqrt{x^2+y^2}$

Let $\varepsilon > 0$, find $\delta = \delta(\varepsilon)$

$\|\vec{x}\| = \sqrt{x^2+y^2} < \delta$, then $|f(x)| < \varepsilon$

$$|f(x)| = \left|\frac{3x^2y}{x^2+y^2}\right| = \frac{3x^2|y|}{x^2+y^2} < \varepsilon$$

$|y| = \sqrt{y^2} \leq \sqrt{x^2+y^2}$, therefore $\left(\frac{\sqrt{y^2}}{\sqrt{x^2+y^2}}\right)^2 \leq 1^2$

Then $\frac{3x^2|y|}{x^2+y^2} = 3|y| \cdot \underbrace{\frac{x^2}{x^2+y^2}}_{\leq 1} \leq 3|y| = 3\sqrt{y^2} \leq 3\underbrace{\sqrt{x^2+y^2}}_{\|\vec{x}\|}$

So, $\underbrace{|f(x)| \leq 3\|\vec{x}\| < 3\delta = \varepsilon}$

then put $\boxed{\delta = \frac{\varepsilon}{3}}$

Proof. Let $\varepsilon > 0$, and suppose $\delta = \frac{\varepsilon}{3}$. Then whenever $\|\vec{x}\| < \delta$,

$$|f(x)| = \frac{3|y|\,x^2}{x^2+y^2} \leq 3|y| \leq 3\|\vec{x}\| < 3\delta = 3\frac{\varepsilon}{3} = \varepsilon.$$

Therefore $\lim_{\vec{x}\to\vec{0}} \frac{3x^2y}{x^2+y^2} = 0$ by the defn. of limit ∎

Defn. A function of multiple variables is said to be <u>continuous</u> at $\vec{x} = \vec{a}$ if and only if

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f\left(\lim_{\vec{x}\to\vec{a}} \vec{x}\right) = f(\vec{a})$$

or

$$\lim_{(x,y)\to(a,b)} f(x,y) = f\left(\lim_{(x,y)\to(a,b)} (x,y)\right) = f(a,b)$$

<u>Examples of continuous functions</u>

<u>Polynomials:</u> $P(\vec{x}) = P(x,y) = 3x^2y^5 + 2xy^4 - 3x^7 + 2y^5 + 1$
cont. on $\mathbb{R}^2$.

<u>Rational:</u> $R(\vec{x}) = \dfrac{P(\vec{x})}{Q(\vec{x})}$ cont. except when $Q(\vec{x}) = 0$.

Trig, logs, exponential, etc. on their domains.

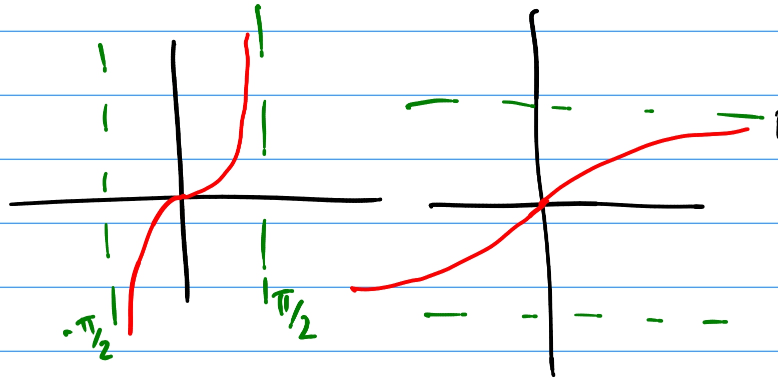
e.g., $T(x,y) = \sin(x+y) + x\sin(y) - \csc(x^2+y^2)$

$\sin\left(\frac{x}{y}\right)$, $y \neq 0$.

<u>Ex.</u> $h(\vec{x}) = \arctan\left(\frac{x}{y}\right)$ $\qquad$ $\text{dom}(\tan^{-1}(u))$: $u \in \mathbb{R}$.

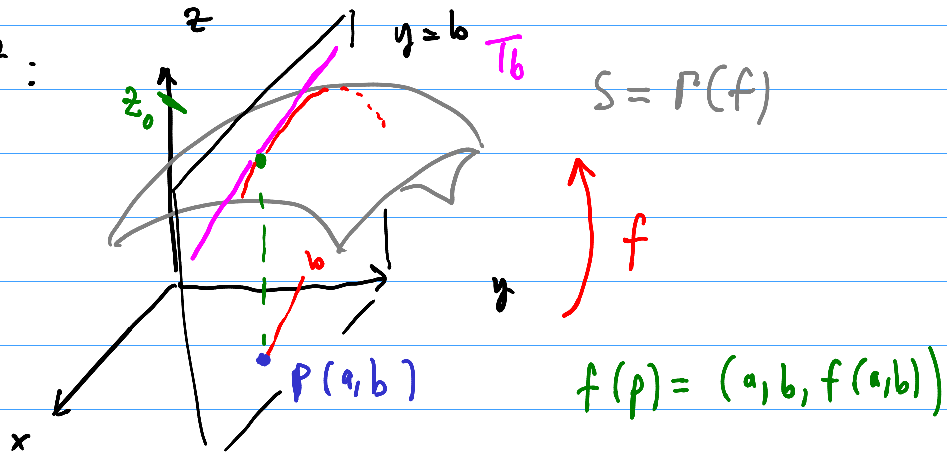
$\Downarrow$

$y \neq 0$



§14.3 – Partial Derivatives

Think in $\mathbb{R}^2$:



consider the plane $y=b$

The partial derivative of f in the x-direction at the point $(a, b, f(a,b))$ is the slope of the tangent line to the intersection curve of $\Gamma(f)$ and $y=b$ at $x=a$.

Notation: $\frac{\partial f}{\partial x}$, $\partial_x f$, f_x, $D_x f$

limit defn:

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a, b+h) - f(a,b)}{h}$$

Idea: In taking ∂_x of f, we regard the y's as constants and take usual derivatives of x.
Similarly for y.

Ex. $f(x,y) = x^3 + x^2y^3 - 2y^2$ Find $\partial_x f$ and $\partial_y f$ at $(2,1)$

$\partial_x f = 3x^2 + 2xy^3$ $\quad \partial_x f(2,1) = 3(4) + 4\cdot 1 = 16$

$\partial_y f = 3x^2y^2 - 4y$ $\qquad$ $\partial_y f(2,1) = 3\cdot4\cdot1 - 4\cdot1 = 8$

Ex. $g(x,y) = x^y$

$$\frac{\partial g}{\partial x} = y\,x^{y-1}$$

$$\frac{\partial g}{\partial y} = \ln(x)\,x^y$$

Ex. $F = \frac{x}{y}$

$$F_x = \frac{1}{y}$$

$$F_y = \frac{-x}{y^2}$$

Ex.

$z = (x^2y - y^3)^5$

$$\frac{\partial z}{\partial x} = 5(x^2y - y^3)^4 \cdot (2xy)$$

$$\frac{\partial z}{\partial y} = 5(x^2y - y^3)^4 \cdot (x^2 - 3y^2)$$

Ex. $f(x,y) = \ln(x^2-3x)\cdot y^2$

$$\frac{\partial f}{\partial x} = y^2 \cdot \frac{2x-3}{x^2-3x}$$

$$\frac{\partial f}{\partial y} = \ln(x^2-3x)\cdot 2y$$

Ex. $x^3 + y^3 + z^3 + 6xyz = 1$ $\qquad$ Find $\frac{\partial z}{\partial x}$ implicitly.

regard $z = z(x,y)$.

$$\frac{\partial}{\partial x}\left(x^3 + y^3 + z^3 + \underbrace{6xy}z = 1\right)$$

$$3x^2 + 0 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\cdot\frac{\partial z}{\partial x} = 0$$

$$(3z^2+6xy)\frac{\partial z}{\partial x} = -3x^2 - 6yz$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-(3x^2+6yz)}{(3z^2+6xy)}}$$

Notation: Second Derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}[f]\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \underbrace{\partial_x \partial_y f}_{\partial_{xy} f} = (f_y)_x = f_{yx}$$

$$\partial_x^2 f = \partial_x \partial_x f = \partial_{xx} f$$

$$\cancel{\partial_x f^2} = (\partial_x f)^2 = \partial_x f \cdot \partial_x f$$

Ex. $f(x,y) = e^x \sin(y)$

$$\frac{\partial^2 f}{\partial x \partial y},\quad \frac{\partial^2 f}{\partial y \partial x},\quad \frac{\partial^2 f}{\partial x^2},\quad \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial f}{\partial x} = e^x \sin y$$

$$\frac{\partial f}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = e^x \sin y$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = e^x \cos y$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = e^x \cos y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -e^x \sin y$$

Thm (Clairaut's Thm) Suppose f is defined on a small disk containing the point (a,b). If $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are both continuous on the disk, then

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(a,b)} = \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(a,b)} .$$
