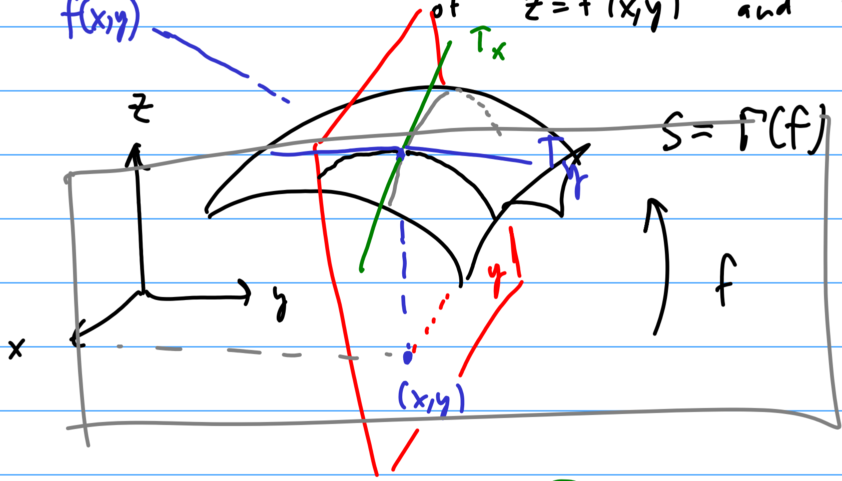
M344 9/18/18

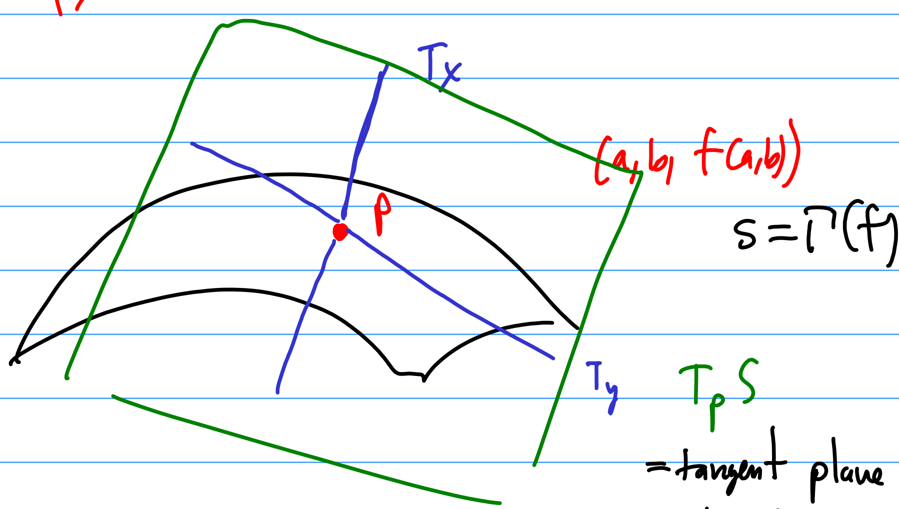
§14.4 – Tangent Planes and Linear Approximations

$z = f(x,y)$

$\frac{\partial f}{\partial x}$ at (a,b): Fix $y=b$, then $f(x,b)$ becomes a calc I function and $\frac{\partial f}{\partial x}(a,b)$ represents the slope of the tan. line to the curve of intersection of $z=f(x,y)$ and $y=b$.



Zooming out:



$= $ tangent plane to S at P.

Our job: Write an equation of T_pS.

Any plane has the form: $Ax + By + Cz + D = 0$

$$\to \quad \frac{A}{C}x + \frac{B}{C}y + z + \frac{D}{C} = 0$$

then solving for z:

$$z = A(x-a) + B(y-b) + C$$

$P(a,b,f(a,b))$

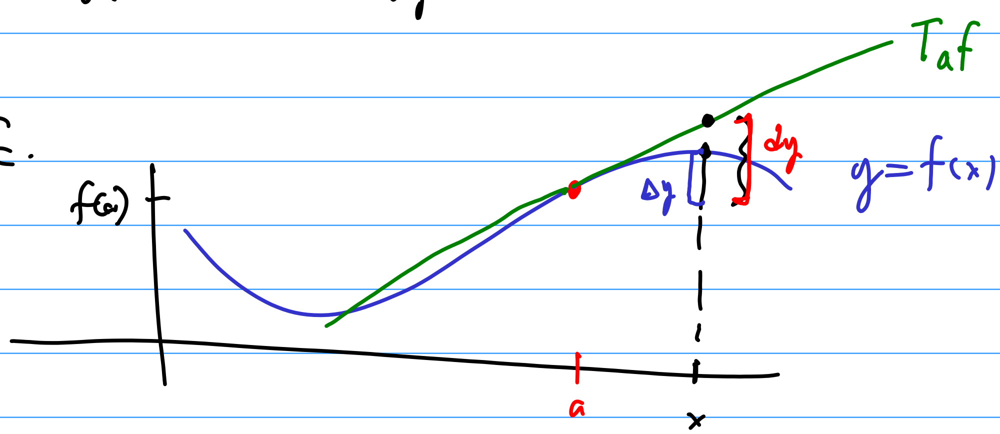
$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ both exist at (a,b).

Then, $\quad T_pS: \boxed{z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)}$

OR

$$\frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) - (z - f(a,b)) = 0$$

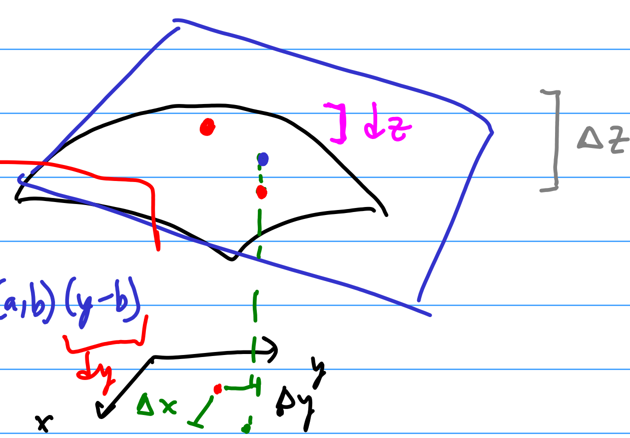
In Calc I.



$$f(x) = f(a) + \Delta y$$

$$f(x) \approx f(a) + dy$$

In Calc III:



dz

T_pS: $z = f(a,b) + \frac{\partial f}{\partial x}(a,b)\underbrace{(x-a)}_{dx} + \frac{\partial f}{\partial y}(a,b)\underbrace{(y-b)}_{dy}$

so $\boxed{dz = \partial_x f\, dx + \partial_y f\, dy}$

Tangent Plane: $z = f(a,b) + \partial_x f (x-a) + \partial_y f (y-b)$

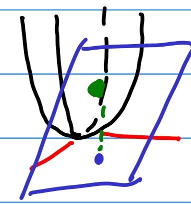
As a function, z is the linearization of f at (a,b)

Differentials: $dz = \partial_x f\, dx + \partial_y f\, dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

$$f(a+dx, b+dy) \approx f(a,b) + dz$$

Ex. $z = 2x^2 + y^2$ Elliptic Paraboloid

$P(1,1)$ Find TpS



TpS: $z = f(P) + \frac{\partial f}{\partial x}(P)(x-a) + \frac{\partial f}{\partial y}(P)(y-b)$

$f(x,y) = z = 2x^2 + y^2$ $\quad$ $f(P) = f(1,1) = 2+1 = 3$

$\frac{\partial f}{\partial x} = 4x$ $\quad$ $\frac{\partial f}{\partial x}(1,1) = 4(1) = 4$

$\frac{\partial f}{\partial y} = 2y$ $\quad$ $\frac{\partial f}{\partial y}(1,1) = 2$

$$\boxed{\text{TpS: } z = 3 + 4(x-1) + 2(y-1)}$$

Approximate $f(1.1, 0.9)$

$$z \approx 3 + 4(1.1-1) + 2(0.9-1)$$
$$= 3 + 4(0.1) + 2(-0.1)$$
$$= 3 + 0.4 - 0.2$$
$$= 3.2$$

$dz = 0.2$

$f(1.1, 0.9) = 2(1.1)^2 + (0.9)^2$
$= 2(1.21) + (0.81)$
$= 2.42 + 0.81$
$= 3.23$

$\Delta z = 0.23$

Ex. $f(x,y) = xe^{xy}$ $\quad P(1,0)$

Find the linearization, $L(x,y)$ (same formula as T_pS)

$L(x,y) = z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ $\quad a=1$, $b=0$

$f(1,0) = 1 \cdot e^{1 \cdot 0} = 1$

$f_x = e^{xy} + xye^{xy}$ $\quad f_x(1,0) = e^0 + 1 \cdot 0 \cdot e^0 = 1$

$f_y = x^2 e^{xy}$ $\quad f_y(1,0) = 1^2 e^0 = 1$

$L(x,y) = 1 + (x-1) + y$

$L(x,y) = x+y$ $\quad$ **NEAR P:** $xe^{xy} \approx x+y$

Ex. $\boxed{1.2\, e^{0.12}} \approx 1.2 + 0.1 = 1.3$

$x = 1.2$ $\quad a(1.2, 0.1)$

$y = 0.1$

xe^{xy}

$1.2\, e^{1.2(0.1)}$

Actual $= 1.35299\ldots$

Ex. $f(x,y) = 1 + x\ln(xy-5)$ Find TpS, or L, at $(2,3)$

$f(2,3) = 1 + 2\cdot\ln(6-5) = 1$

$\partial_x f = \ln(xy-5) + x\cdot\frac{y}{xy-5}$ $\partial_x f(2,3) = \ln 1 + \frac{6}{1} = 6$

$\partial_y f = \frac{x^2}{xy-5}$ $\partial_y f(2,3) = \frac{4}{1} = 4$

$$L(x,y) = z = f(2,3) + \partial_x f(2,3)(x-2) + \partial_y f(2,3)(y-3)$$
$$= 1 + 6(x-2) + 4(y-3)$$
$$\boxed{z = 6x + 4y - 23}$$

Ex. $f(x,y) = e^{-xy}\cos y$ Find dz

$$dz = \partial_x f\, dx + \partial_y f\, dy \leftarrow$$

$\partial_x f = -y e^{-xy}\cos y$

$\partial_y f = -x e^{-xy}\cos y - e^{-xy}\sin y$

$$\boxed{dz = -y e^{-xy}\cos y\, dx - \left(x e^{-xy}\cos y + e^{-xy}\sin y\right) dy}$$

§14.5 – Chain Rule

Case I. $f(x,y)=z$ suppose $x=x(t)$ and $y=y(t)$.

$z=f(x(t),y(t))$ may be regarded as $z(t)$

$$\frac{dz}{dt}=?$$

$$\frac{dz}{dt}=\underbrace{\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy}_{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

"Chain Rule in each slot!"

Ex. $z=x^2y+3xy^4$ $\quad x=\sin(2t)$ $\quad dx/dt=2\cos(2t)$

$y=\cos t$ $\quad dy/dt=-\sin t$

Find: $\dfrac{dz}{dt}=\dfrac{\partial z}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$

$\dfrac{\partial z}{\partial x}=2xy+3y^4$

$\dfrac{\partial z}{\partial y}=x^2+12xy^3$

$$\boxed{\frac{dz}{dt}=(2xy+3y^4)\,2\cos(2t)+(x^2+12xy^3)(-\sin t)}$$