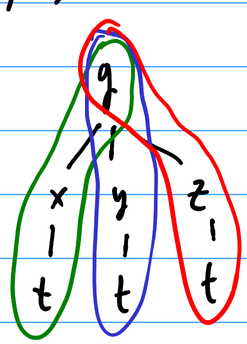
M344 9/20/18

§14.5 – Chain Rule

$f(x,y)$ $\quad x = x(t),\ y(t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$g(x,y,z)$ $\quad x = x(t),\ y = y(t),\ z(t)$

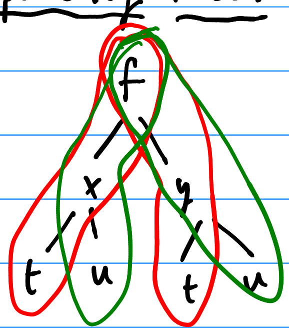


$$\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt}$$

Consider: $f(x,y)$ $\quad x = x(t,u)$ $\quad y = y(t,u)$

$$f(x(t,u), y(t,u)) = f(t,u)$$

Dependency Tree:

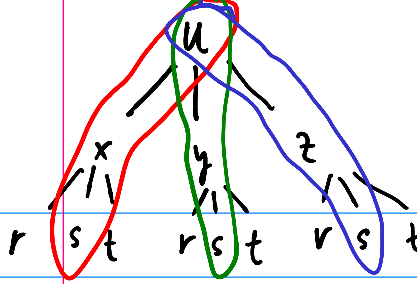


$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

Ex. $u = x^4 y + z^3 y^2$ $\quad x = rse^t$ $\quad y = rs^2e^{-t}$ $\quad z = r^2 s \sin t$

$$\frac{\partial u}{\partial s} \text{ at } (r,s,t) = (2,1,0)$$



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial s}} + \frac{\partial u}{\partial y}\underbrace{\frac{\partial y}{\partial s}} + \underbrace{\frac{\partial u}{\partial z}}_{=0}\underbrace{\frac{\partial z}{\partial s}}$$

$x(2,1,0) = 2\cdot 1\cdot e^{0} = 2$ $\qquad$ $z(2,1,0) = 2^2\cdot 1\cdot \sin 0 = 0$

$y(2,1,0) = 2\cdot 1^2\cdot e^{-0} = 2$

$u = x^4 y + z^3 y^2$

$\frac{\partial u}{\partial x} = 4x^3 y \xrightarrow{P} 4\cdot 2^3\cdot 2 = 64$

$\frac{\partial u}{\partial y} = x^4 + 2z^3 y \to 2^4 + 2\cdot 0^3\cdot 2 = 16$

$\frac{\partial u}{\partial z} = 3z^2 y^2 \to 3\cdot 0^2\cdot 2^2 = 0$

$\frac{\partial x}{\partial s} = re^{t} \xrightarrow{P} 2\cdot e^{0} = 2$

$\frac{\partial y}{\partial s} = 2rse^{-t} \to 2\cdot 2\cdot 1\cdot e^{-0} = 4$

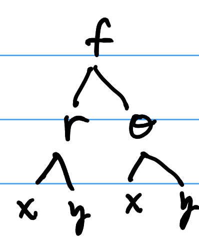
$$\left.\frac{\partial u}{\partial s}\right|_{(2,1,0)} = 64\cdot 2 + 16\cdot 4 + 0\cdot\frac{\partial z}{\partial s} = 192$$

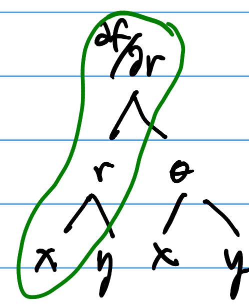
Ex. Let $f = f(x,y)$.

$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ $\qquad$ $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\}$ $\qquad$ Δf = "Laplacian of f"

Ex: write out Δf in Polar coord's.

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan(y/x) \end{cases}$$





$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\underbrace{\frac{\partial f}{\partial r}\frac{\partial r}{\partial x}} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}\right)$$

$$= \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r}\right)}\cdot\frac{\partial r}{\partial x} + \underbrace{\frac{\partial f}{\partial r}\cdot\frac{\partial^2 r}{\partial x^2}} + \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \theta}\right)}\frac{\partial \theta}{\partial x} + \underbrace{\frac{\partial f}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}}$$

$$= \left[\frac{\partial^2 f}{\partial r^2}\cdot\frac{\partial r}{\partial x} + \frac{\partial^2 f}{\partial\theta\partial r}\frac{\partial\theta}{\partial x}\right]\frac{\partial r}{\partial x} + \frac{\partial f}{\partial r}\cdot\frac{\partial^2 r}{\partial x^2} + \left[\frac{\partial^2 f}{\partial r\partial\theta}\frac{\partial r}{\partial x} + \frac{\partial^2 f}{\partial\theta^2}\frac{\partial\theta}{\partial x}\right]\frac{\partial\theta}{\partial x} + \frac{\partial f}{\partial\theta}\frac{\partial^2\theta}{\partial x^2}$$

$$r = \sqrt{x^2+y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$

$$(\tan^{-1}(u))' = \frac{u'}{1+u^2}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\sqrt{x^2+y^2} - x\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{y^2}{\sqrt{x^2+y^2}^{\,3}} = \frac{y^2}{r^3}$$

$$\frac{\partial \theta}{\partial x} = \frac{1 \cdot {-y}/{x^2}}{1+({y}/{x})^2} = \frac{-y}{x^2+y^2} = \frac{-y}{r^2}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2} = \frac{2xy}{r^4}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial r^2}\left(\frac{\partial x}{\partial r}\right)^2 + \frac{\partial^2 f}{\partial \theta \partial r}\frac{\partial \theta}{\partial x}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 f}{\partial r \partial \theta}\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial^2 f}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{\partial f}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}$$

$$= \frac{\partial^2 f}{\partial r^2}\left(\frac{x}{r}\right)^2 + 2\frac{\partial^2 f}{\partial \theta \partial r}\frac{x}{r} \cdot \frac{-y}{r^2} + \frac{\partial^2 f}{\partial \theta^2}\left(\frac{-y}{r^2}\right)^2 + \frac{\partial f}{\partial r}\left(\frac{y^2}{r^3}\right) + \frac{\partial f}{\partial \theta}\left(\frac{2xy}{r^4}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{x^2}{r^2}\frac{\partial^2 f}{\partial r^2} - \frac{2xy}{r^3}\frac{\partial^2 f}{\partial \theta \partial r} + \frac{y^2}{r^4}\frac{\partial^2 f}{\partial \theta^2} + \frac{y^2}{r^3}\frac{\partial f}{\partial r} + \frac{2xy}{r^4}\frac{\partial f}{\partial \theta}$$

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$$+ \quad \frac{\partial^2 f}{\partial y^2} = \frac{y^2}{r^2}\frac{\partial^2 f}{\partial r^2} + \cdots$$

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

Implicit Differentiation

Consider Calc I: e.g., $\underbrace{x^2+y^2}_{F(x,y)} = r^2$ assume $y = y(x)$

$$F(x,y) = F(x, y(x))$$

and

$$\frac{dF}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

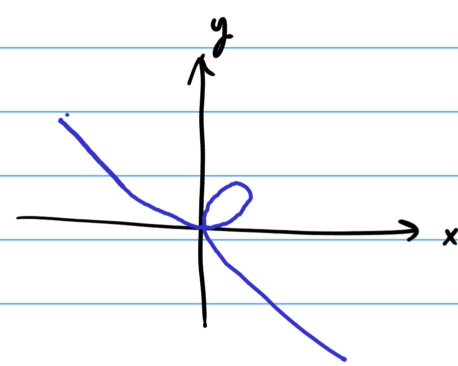
so $\frac{dF}{dx} = 0 \Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y} = \frac{-\partial_x F}{\partial_y F}$$

Circle : $x^2+y^2 = r^2$ Find $dy/dx = \frac{-\partial_x F}{\partial_y F}$ where $F = x^2+y^2$

$$\left.\begin{aligned} \partial_x F &= 2x \\ \partial_y F &= 2y \end{aligned}\right\} \boxed{\frac{dy}{dx} = \frac{-x}{y}} \quad \ast$$

Ex. Folium of Descartes: $x^3+y^3 = 6xy$



Find $\frac{dy}{dx}$. $F(x,y) = x^3+y^3-6xy$

Then $\frac{dy}{dx} = \frac{-\partial_x F}{\partial_y F} = \frac{-(3x^2-6y)}{3y^2-6x}$

$$\boxed{\frac{dy}{dx} = \frac{x^2-2y}{2x-y^2}}$$

We can extend this idea to $F(x,y,z) = C$

$$\frac{\partial z}{\partial x} = \frac{-\partial_x F}{\partial_z F} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-\partial_y F}{\partial_z F}$$

Ex. $\underbrace{x^3 + y^3 + z^3 + 6xyz}$ $= 1$

$F(x,y,z) = x^3 + y^3 + z^3 + 6xyz$

Find: $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial x} = \frac{-\partial_x F}{\partial_z F} = \frac{-(3x^2 + 6yz)}{3z^2 + 6xy}$$