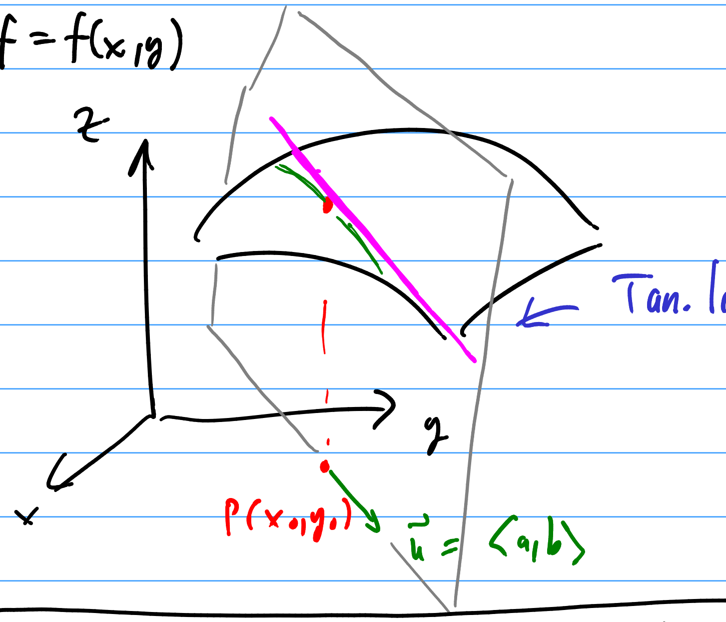
M344 9/25/18

§14.6 – Directional Derivatives

$z = f = f(x,y)$



$S = \Gamma(f)$

$\|\vec{u}\| = 1$

slope = ? = Directional derivative of f at (x_0,y_0) in the direction of $\vec{u}$.

(*) $$\boxed{D_{\vec{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ah,\, y_0+bh) - f(x_0,y_0)}{h}}$$

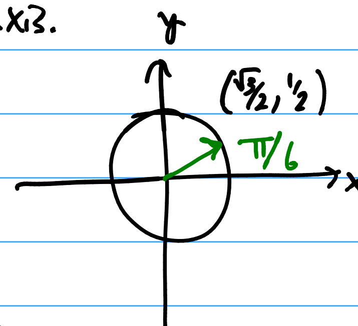
To compute: $\boxed{D_{\vec{u}} f(p) = \partial_x f(p)\cdot a + \partial_y f(p)\cdot b}$ Thm.

Proof: RE.

Ex. $f(x,y) = x^3 - 3xy + 4y^2$

$\vec{u}$ makes an angle of $\pi/6$ w/ positive x-axis.

$p(1,2)$



$D_{\vec{u}} f(P) = \partial_x f(p)\cdot a + \partial_y f(p)\cdot b$

$\partial_x f = 3x^2 - 3y$ $\quad \partial_x f(P) = \partial_x f(1,2) = 3\cdot 1^2 - 3\cdot 2 = -3$

$\partial_y f = -3x + 8y$ $\quad \partial_y f(P) = \partial_y f(1,2) = -3\cdot 1 + 8\cdot 2 = 13$

So $\vec{u} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$ and

$$D_{\vec{u}} f(P) = \underbrace{\partial_x f(P) \cdot a + \partial_y f(P) \cdot b}_{} = -3 \cdot \frac{\sqrt{3}}{2} + 13 \cdot \frac{1}{2} = \frac{13 - 3\sqrt{3}}{2}$$

$$D_{\vec{u}} f(P) = \partial_x f(P) \cdot a + \partial_y f(P) \cdot b \qquad \vec{u} = \langle a, b \rangle$$

Def'n. Let f be a smooth function of x, y. The gradient of f is a vector-valued function

$$\nabla f = \langle \partial_x f, \partial_y f \rangle$$

and

$$\nabla f(P) = \langle \partial_x f(P), \partial_y f(P) \rangle$$

Thm. $D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u}$, $\quad \|\vec{u}\| = 1$.

Ex. $f(x,y) = x^2 y^3 - 4y$

Find $\begin{cases} \nabla f \\ \nabla f(2,-1) \\ D_{\vec{v}} f(2,-1) \text{ where } \boxed{\vec{v} = 2\vec{i} + 5\vec{j}} \end{cases}$

$$\nabla f = \langle \partial_x f, \partial_y f \rangle$$

$$\nabla f = \langle 2xy^3, 3x^2y^2 - 4 \rangle$$

$$\nabla f(2,-1) = \langle 2(2)(-1)^3, 3(2^2)(-1)^2 - 4 \rangle = \langle -4, 8 \rangle$$

$D_{\vec{v}} f(P) = \nabla f(P) \cdot \vec{v}$? No! $= \nabla f(P) \cdot \vec{u}$ where $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$

$\|\vec{v}\| = \sqrt{2^2 + 5^2} = \sqrt{29}$ so $\vec{u} = \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \rangle$

and $D_{\vec{v}} f(P) = D_{\vec{u}} f(P) = \nabla f(2,-1) \cdot \vec{u} = \langle -4, 8 \rangle \cdot \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \rangle$

$$= \frac{-8 + 40}{\sqrt{29}} = \frac{32}{\sqrt{29}}.$$

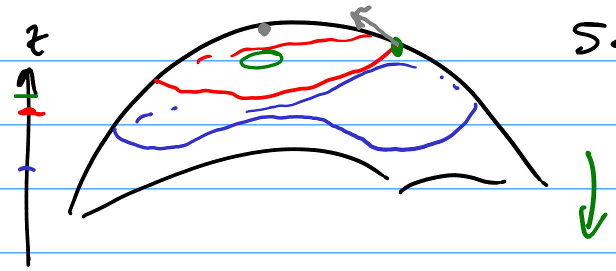
Ex. $f(x,y) = \sin x + e^{xy}$ find ∇f, $\nabla f(0,1)$

$$\nabla f = \langle \partial_x f, \partial_y f \rangle = \langle \cos x + y e^{xy}, x e^{xy} \rangle$$

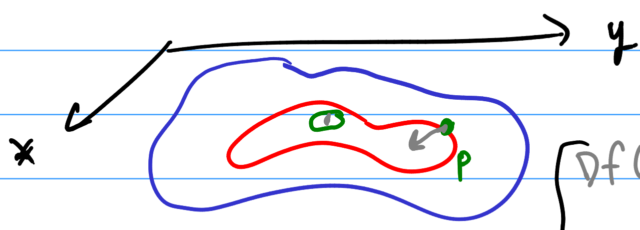
$$\nabla f(0,1) = \langle \cos 0 + 1 \cdot e^0, 0 \cdot e^0 \rangle = \langle 2, 0 \rangle$$

Geometric Interpretation

$f : (x,y) \longmapsto z$



$S = \Gamma(f)$



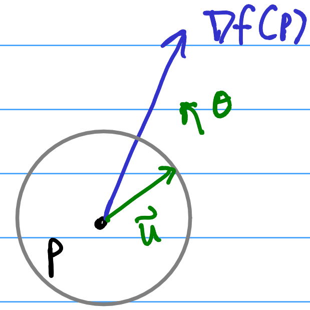
contours or level curves.

$\nabla f(P)$ points in the direction of steepest ascent.

$\nabla f(P)$ is orthogonal to the contour line through P.

Thm. Let f be a smooth function of at least 2 variables. Then the maximum value attain by $D_{\vec{u}} f$ at any pt P is $\|\nabla f(P)\|$, and this occurs when $\vec{u}$ is in the same direction as $\nabla f(P)$.

Proof. Consider the picture



$$D_{\vec u} f(P) = \nabla f(P)\cdot \vec u = \|\nabla f(P)\| \overbrace{\|\vec u\|}^{1} \cos\theta$$
$$= \|\nabla f(P)\| \cos\theta$$

This is max. when $\cos\theta = 1$, whence

$$\max_{\vec u}\left(D_{\vec u} f(P)\right) = \|\nabla f(P)\|.$$

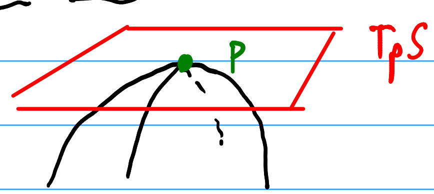
This occurs when $\theta = 0$, so $\vec u$ is in the same direction as $\nabla f(P)$. ∎

<u>Defn</u>. A <u>critical point</u> of $f(x,y)$ is a point where either

$$\begin{cases} \nabla f(P) = \vec 0 \\ \text{or } \nabla f(P) = \text{undef.} \end{cases}$$

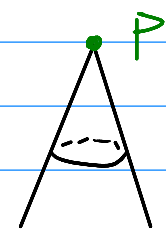
<u>Local</u> <u>Max</u>: zoom in:

$\nabla f(P) = \vec 0$

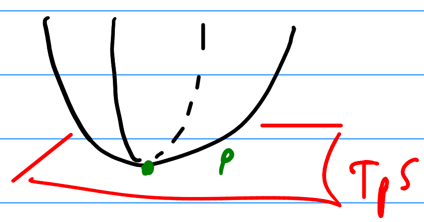


surface is always <u>below</u> tan. plane near P.

$\nabla f(P) = \text{undef.}$:

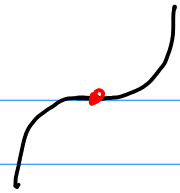


<u>Local</u> <u>Min</u>: $\nabla f(P) = \vec 0$

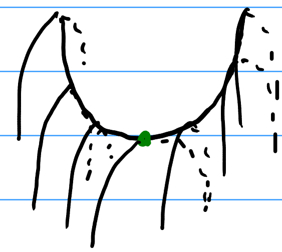


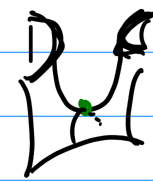
Tan. plane is below $\Gamma(f)$.

Calc I:



Neither: $\nabla f(P) = \vec{0}$ but not a max/min





hyperbolic paraboloid

"saddle point"

To check if a critical pt is a local max/min or saddle pt, we use a Second Derivative Test.

We build the Hessian of f.

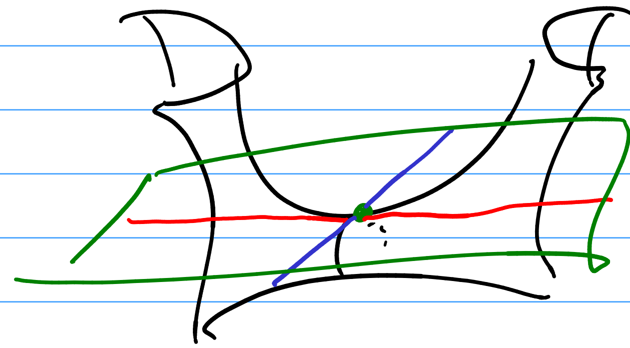
$$\hat{H}_f = \hat{H} = \begin{pmatrix} \partial_{xx} f & \partial_{xy} f \\ \partial_{yx} f & \partial_{yy} f \end{pmatrix}$$

$$H = \det(\hat{H}) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$$

Thm (SDT). Let f be a twice diff'able function whose 2nd partials are continuous. And let P be a critical pt: $\nabla f(P) = \vec{0}$. Then,

1. If $H(P) > 0$ and $\partial^2 f / \partial x^2 (P) > 0$, then P is local min.
2. If $H(P) > 0$ and $\partial^2 f / \partial x^2 (P) < 0$, then P is a local max.
3. If $H(P) < 0$, then P is a saddle pt.

At a saddle pt:



Ex. $f(x,y) = x^4 + y^4 - 4xy + 1$ Classify all CPs.

$\nabla f = \vec{0}$ to find CPs.

$$\left.\begin{aligned} f_x &= 4x^3 - 4y = 0 \\ f_y &= 4y^3 - 4x = 0 \end{aligned}\right\}$$

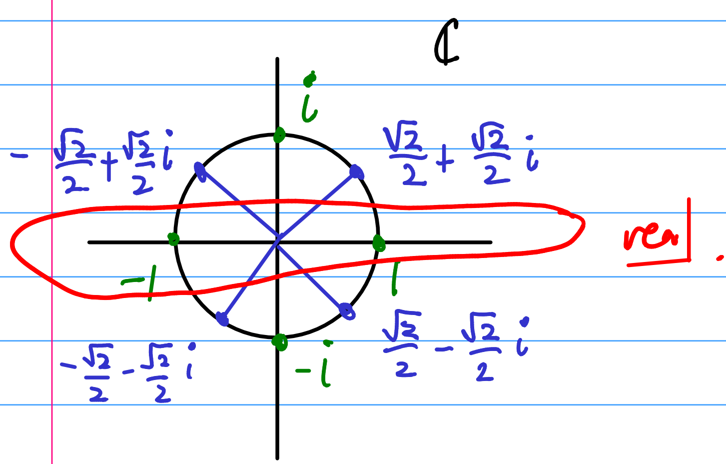
$y = x^3$

$x = y^3$

$x = (x^3)^3$

$x^9 - x = 0$

$x(\underbrace{x^8 - 1}) = 0$



$x^8 = 1$

$x = \sqrt[8]{1} = \pm 1$ and 6 others.

$x = 0, 1, -1$ taken in pairs

$y = 0, 1, -1$

$(0,0)$

$(1,1)$

and $(-1,-1)$

$(1,1)$: $\hat{H} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} \boxed{12x^2} & -4 \\ -4 & 12y^2 \end{pmatrix}$

$H = 144x^2y^2 - 16$

$$\left.\begin{aligned} H(1,1) &= 144-16 > 0 \\ f_{xx}(1,1) &= 12\cdot 1^2 = 12 > 0 \end{aligned}\right\} \quad (1,1) \text{ is a local min}$$

$H(0,0) = 0-16 < 0$ saddle pt.