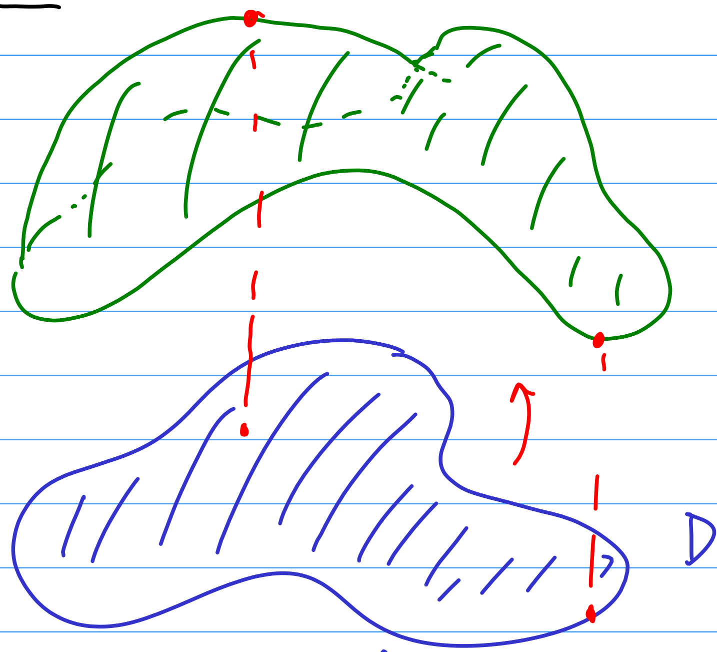
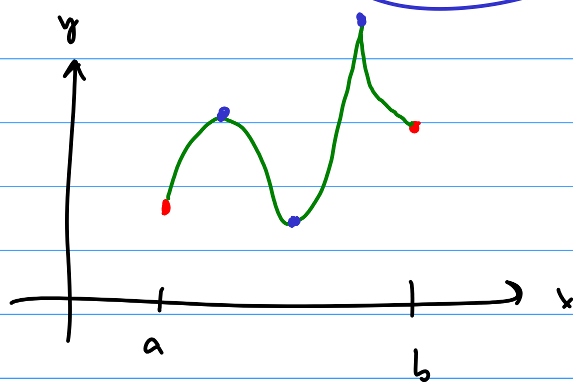
M344 9/27/18

Thm. If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute max and absolute min on D.



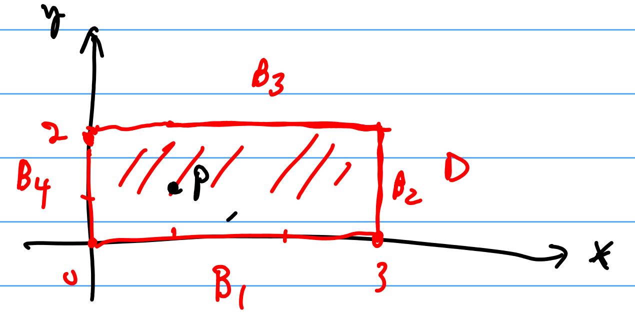
Calc I.



Check: interior CPs
boundary values.

Ex. $f(x,y) = x^2 - 2xy + 2y$ on $D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$

Find max/min values of f on D.



Start w/ interior: Find CP.

B_1: $y = 0$: $0 \le x \le 3$

$$\nabla f = \langle \partial_x f, \partial_y f \rangle = \langle 2x-2y, 2-2x \rangle = \vec{0}$$

$$\begin{cases} 2x-2y=0 & \rightarrow & 2-2y=0 \Rightarrow y=1 \\ 2-2x=0 & \rightarrow & 2=2x \Rightarrow x=1 \end{cases}$$

CP: $(1,1)$ is in D.

plug in: $f(1,1) = 1^2 - 2(1)(1) + 2(1) = 1$

Now, boundary:

B_1: $y=0$ $\quad f(x,0) = x^2 - 2x\cdot 0 + 2\cdot 0 = x^2$ $\quad 0 \le x \le 3$

$f(x,0) = x^2$ $\quad f'(x,0) = 2x$ $\quad$ CP: $x=0$.

$f(0,0) = 0$

$f(3,0) = 9$

B_2: $x=3$ $\quad f(3,y) = 9 - 2\cdot 3\cdot y + 2y = 9-4y$ $\quad 0 \le y \le 2$

$f(3,0) = 9$

$f(3,2) = 9-8 = 1$

B_3: $y=2$ $\quad f(x,2) = x^2 - 2\cdot 2\cdot x + 2\cdot 2 = x^2 - 4x + 4 = (x-2)^2$

$0 \le x \le 3$

CP: $f(2,2) = (2-2)^2 = 0$

$f(3,2) = (3-2)^2 = 1^2 = 1$

$f(0,2) = (-2)^2 = 4$

B_4: $x=0$ $\quad f(0,y) = 0^2 - 2\cdot 0\cdot y + 2y = 2y$ $\quad$ No CPs ✓

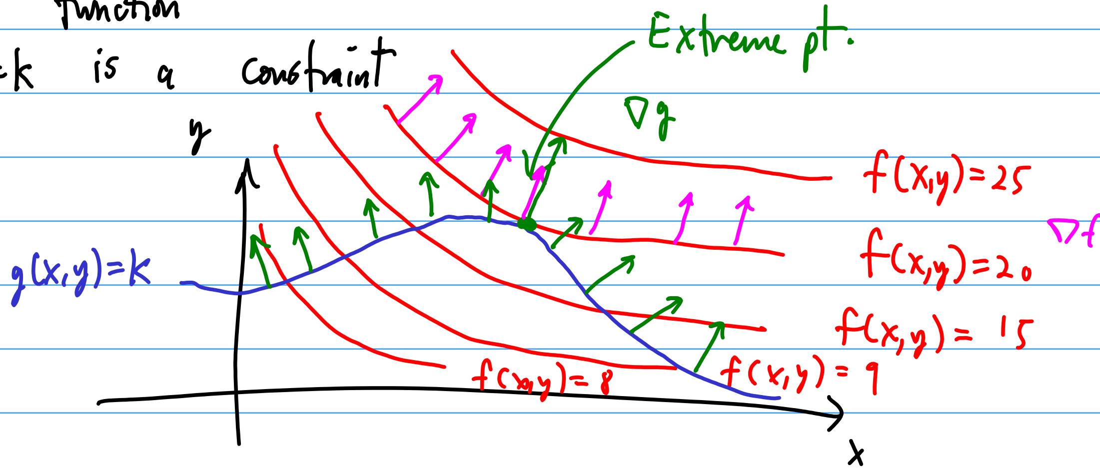
Abs. Max. of f on D is 9.

Abs. min. value of f on D is 0.

§14.8 ~ Lagrange Multipliers

$f(x,y)$ function

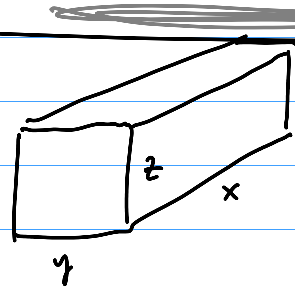
$g(x,y)=k$ is a constraint



At the extreme pt ∇f is parallel to ∇g: $\nabla f = \lambda \nabla g$ for some $\lambda \neq 0$.

Ex.

$\underbrace{x,y,z>0}$



Open Top:

$SA = 20 \text{ units}^2$

Find the dimensions that maximize volume.

$f(x,y,z)$ = volume

$f(x,y,z) = xyz$

$g(x,y,z)$ = surface area

$g(x,y,z) = xy + 2zy + 2zx = 20$

Want $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ solve for x,y,z.

$\nabla f = \langle yz, xz, xy \rangle$

$\nabla g = \langle y+2z, x+2z, 2y+2x \rangle$

Lagrange: $\lambda \nabla f = \nabla g$:
$$\begin{cases} \lambda yz = y+2z \\ \lambda xz = x+2z \\ \lambda xy = 2x+2y \end{cases}$$

$\underbrace{xy + 2xz + 2yz = 20}$

$$\lambda = \frac{y+2z}{yz} = \frac{x+2z}{xz} = \frac{2x+2y}{xy}$$

$$yz(x+2z) = xz(y+2z)$$

$$\cancel{xyz} + \cancel{2}y\cancel{z^2} = \cancel{xyz} + \cancel{2}x\cancel{z^2}$$

$$\boxed{y = x}$$

$$\frac{x+2z}{xz} = \frac{4x}{x^2}$$

$$x^3 + 2zx^2 = 4x^2z$$

$$(4x^2z - 2zx^2) = x^3$$

$$2zx^2 = x^3$$

$$\boxed{z = \frac{1}{2}x}$$

$$xy + 2xz + 2yz = 20$$

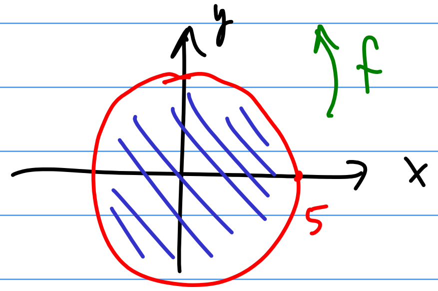
$$x^2 + x^2 + x^2 = 20$$

$$3x^2 = 20$$

$$x = \pm\sqrt{\frac{20}{3}} = \sqrt{\frac{20}{3}} \qquad y = \sqrt{\frac{20}{3}} \qquad z = \frac{1}{2}\sqrt{\frac{20}{3}}$$

Ex. $f(x,y) = x^2 - 4x + y^2 - 9y$ $\quad D$: disk: $x^2+y^2 \le 25$

Find the Abs. max/min of f on D.



Interior: $\nabla f = \vec{0}$

$$\nabla f = \langle 2x-4,\ 2y-9 \rangle = \langle 0, 0 \rangle = \begin{cases} 2x-4=0 & x=2 \\ 2y-9=0 & y=9/2 \end{cases}$$

$P(2, 9/2)$ is in D

$$\sqrt{2^2 + \left(\frac{9}{2}\right)^2} = \sqrt{4 + \frac{81}{4}} = \sqrt{\frac{97}{4}} \le \sqrt{\frac{100}{4}} = 5$$

$$f(2, 9/2) = 2^2 - 4(2) + \left(\tfrac{9}{2}\right)^2 - 9\left(\tfrac{9}{2}\right) = 4 - 8 + \frac{81}{4} - \frac{81}{2}$$

$$= -4 - \frac{81}{4} = -\frac{97}{4}$$

Boundary: $x^2 + y^2 = 25$ $g(x,y) = x^2 + y^2$ $\nabla g = \langle 2x, 2y \rangle$

$\nabla f = \langle 2x-4, 2y-9 \rangle$

$\nabla f = \lambda \nabla g$:

$$\begin{cases} 2x - 4 = 2\lambda x \\ 2y - 9 = 2\lambda y \\ x^2 + y^2 = 25 \end{cases}$$

$$\lambda = \frac{2x-4}{2x} = \frac{2y-9}{2y} \Rightarrow \cancel{4xy} - 8y = \cancel{4xy} - 18x$$

$$y = \frac{18}{8}x = \frac{9}{4}x$$

$$x^2 + \left(\tfrac{9}{4}x\right)^2 = 25$$

$$\left(1 + \frac{81}{16}\right)x^2 = 25$$

$$x^2 = \frac{16 \cdot 25}{97} \qquad x = \pm\frac{20}{\sqrt{97}}$$

$$y = \pm\frac{45}{\sqrt{97}}$$

Boundary pts to be checked are $\left(\frac{20}{\sqrt{97}}, \frac{45}{\sqrt{97}}\right)$ and $\left(\frac{-20}{\sqrt{97}}, \frac{-45}{\sqrt{97}}\right)$

You can do it ☺