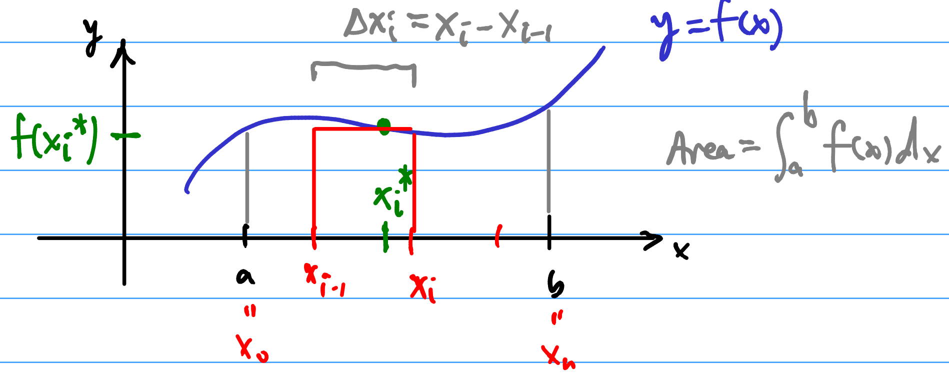
M344 – Calc III 10/2/18

Ch 15 – Multiple Integrals

§15.1-2 : Double Integrals

Recall Calc I: $y=f(x)$



$$\Delta A_i = f(x_i^*)\Delta x_i$$

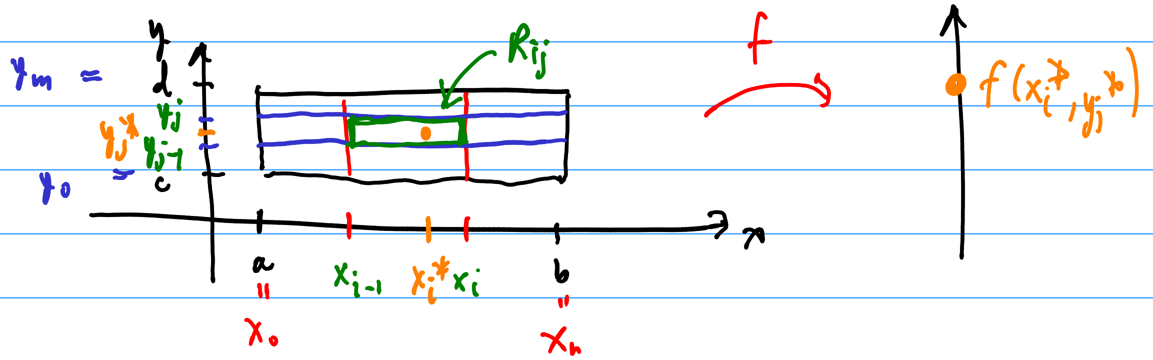
$$\text{Area} \approx \sum_{i=1}^{n} \Delta A_i = \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

and

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = \int_a^b f(x)\,dx$$

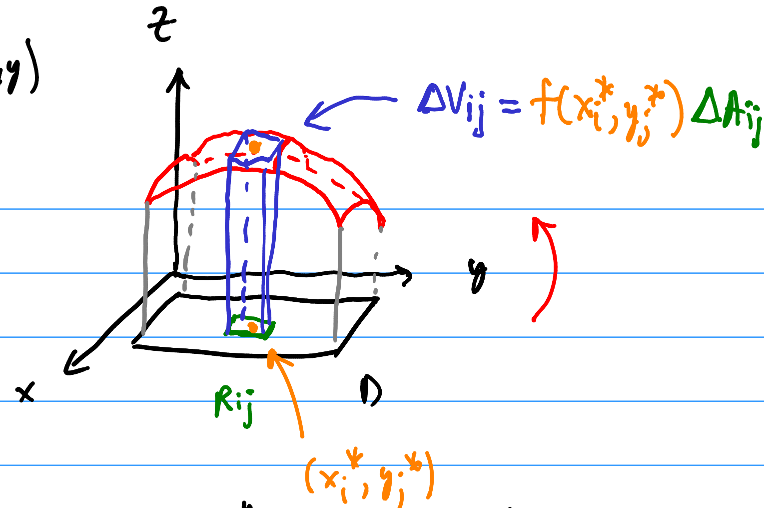
Now, Calc III: $z = f(x,y)$

First attempt: start w/ rectangles in domain



Zoom out. $z = f(x,y)$

Volume $= \iint_D f(x,y)\, dA$



In general, Volume $\approx \sum_{j=1}^{m} \sum_{i=1}^{n} \Delta V_{ij} = \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*)\, \Delta A_{ij}$

Defn: $\iint_D f(x,y)\, dA = \lim_{m\to\infty} \sum_{j=1}^{m} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_j^*)\, \Delta A_{ij}$

For rectangles, $\Delta A_{ij} = \text{Area}(R_{ij}) = \Delta x_i\, \Delta y_j$

We obtain,

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

The last two are iterated integrals:

$$\int_a^b \left[\int_c^d f(x,y)\, dy \right] dx$$

Ex. $\iint_R (x - 3y^2)\, dA$, $R = [0,2] \times [1,2]$
$= \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}$

$$\int_0^2 \left[\int_1^2 (x - 3y^2)\, dy \right] dx = \int_0^2 \left[xy - y^3 \Big|_{y=1}^{2} \right] dx$$

$$= \int_0^2 \left[2x - 8 - (x - 1) \right] dx$$

$$= \int_0^2 x - 7\, dx$$

$$= \tfrac{1}{2}x^2 - 7x \Big|_{x=0}^{2} = 2 - 14 = -12$$

$$\int_1^2 \left[\int_0^2 x - 3y^2 \, dx \right] dy = \int_1^2 \left[\frac{1}{2}x^2 - 3xy^2 \Big|_{x=0}^{2} \right] dy$$

$$= \int_1^2 2 - 6y^2 \, dy$$

$$= 2y - 2y^3 \Big|_{y=1}^{2} = 4 - 16 - (2-2) = -12$$

Fubini's Theorem. If f is a continuous function on the rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

Ex. $\iint_R y \sin(xy)\, dA \qquad R = [1,2] \times [0,\pi]$

continuous ✓

$$\int_1^2 \left[\int_0^\pi y \sin(xy)\, dy \right] dx \overset{\text{OR}}{=} \int_0^\pi \left[\int_1^2 y \sin(xy)\, dx \right] dy$$

$$= \int_0^\pi y \underbrace{\int_1^2 \sin(xy)\, dx}\; dy$$

$$= \int_0^\pi y \left[-\frac{1}{y} \cos(xy) \Big|_{x=1}^{2} \right] dy$$

$$= \int_0^\pi \cancel{y} \left[-\frac{1}{\cancel{y}} \cos(2y) + \frac{1}{\cancel{y}} \cos y \right] dy$$

$$= \int_0^\pi \cos y - \cos(2y)\, dy$$

$$= \left[\sin y - \frac{1}{2} \sin(2y) \right] \Big|_0^\pi = 0.$$

Ex. $\iint_R \sin x \cos y \, dA \qquad R = [0, \pi/2] \times [0, \pi/2]$

$$\iint_R \sin x \cos y \, dA = \int_0^{\pi/2} \int_0^{\pi/2} \sin x \boxed{\cos y} \, dx \, dy$$

$$= \int_0^{\pi/2} \cos y \left[\int_0^{\pi/2} \sin x \, dx \right] dy$$

$$= \int_0^{\pi/2} \cos y \, dy \cdot \int_0^{\pi/2} \sin x \, dx \qquad \text{"separable"}$$

$$= \left[\sin y \Big|_0^{\pi/2} \right] \cdot \left[-\cos x \Big|_0^{\pi/2} \right] = [1 - 0] \cdot [-0 + 1]$$

$$= 1 \cdot 1 = 1$$

Ex. $\iint_R xy e^{x^2 y} \, dA \qquad R = [0,1] \times [0,2]$

$$= \int_0^2 \frac{1}{2} \left[\int_0^1 2xy e^{x^2 y} \, dx \right] dy$$

$u = x^2 y \qquad u(1) = y$

$\frac{\partial u}{\partial x} = 2xy \qquad u(0) = 0$

$du = 2xy \, dx$

$$= \frac{1}{2} \int_0^2 \left[\int_0^y e^u \, du \right] dy$$

$$= \frac{1}{2} \int_0^2 e^y - 1 \, dy$$

$$= \frac{1}{2} (e^y - y) \Big|_{y=0}^{2} = \frac{1}{2} \left(e^2 - 2 - (e^0 - 0) \right) = \frac{1}{2} (e^2 - 3)$$

Ex. $\iint_R \frac{x}{1+xy} \, dA \qquad R = [0,1] \times [0,1]$

$$= \int_0^1 \int_0^1 \frac{x}{1+xy} \, dy \, dx$$

$u = 1 + xy \qquad u(1) = 1 + x$

$\frac{\partial u}{\partial y} = x \qquad u(0) = 1$

$du = x \, dy$

$$= \int_0^1 \left[\int_1^{1+x} \frac{1}{u} \, du \right] dx$$

$$= \int_0^1 \ln(1+x) \, dx$$

$u = 1 + x \qquad u(1) = 2$

$du = dx \qquad u(0) = 1$

Nick
Giaquinto

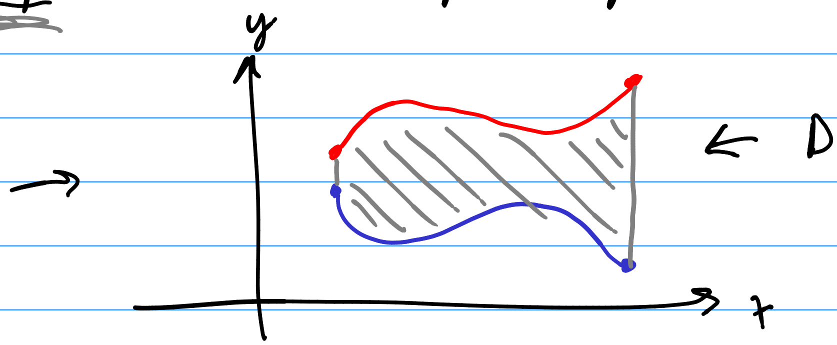
$$= \int_1^2 \underbrace{\ln(u)}\, du = u\ln(u) - \int \frac{1}{u}\cdot u\, du \Big|_1^2 = u\ln u - u \Big|_1^2$$

$U = \ln(u) \qquad dV = du$

$dU = \frac{1}{u}du \qquad V = u$

$$= 2\ln 2 - 2 - (1\cancel{\ln 1}^{\,0} - 1) = \boxed{2\ln 2 - 1}$$

Next step: What about more general regions?



$z = f(x, y)$

$\nabla = \langle f_x, f_y \rangle$