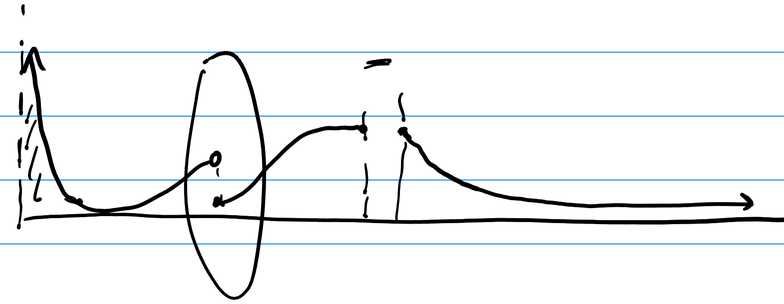
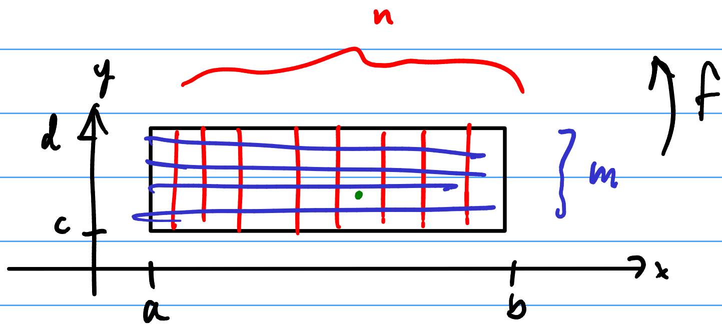
M344 10/11/18

Ch15 - Integration

$f(x,y)$ integrable function



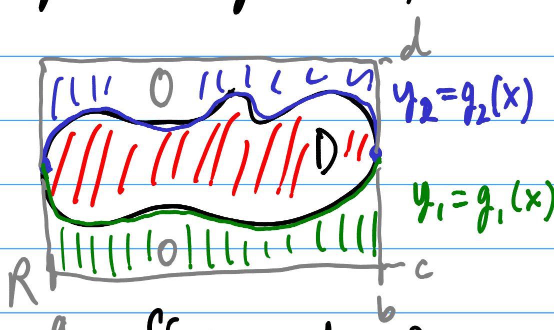


$$V = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*)\, \Delta A_{ij}$$

Fubini's Thm: $V = \iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$

when f is continuous.

§15.3 Integrals over general regions



$f(x,y)$ defined and continuous on D

$\iint_D f(x,y)\, dA = ?$

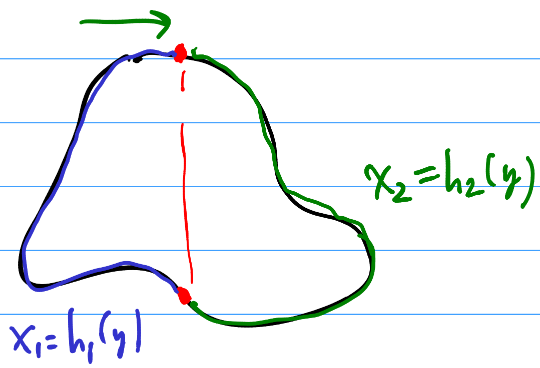
let $F(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases}$

$$\underbrace{\iint_R F(x,y)\, dA} = \cancel{\int_a^b \int_c^{y_1} 0\, dA} + \underbrace{\int_a^b \int_{y_1}^{y_2} f(x,y)\, dy\, dx} + \cancel{\int_a^b \int_{y_2}^{d} 0\, dA}$$

So, $V = \int_a^b \int_{y_1}^{y_2} f(x,y)\,dy\,dx$ where y_1, y_2 are functions of x.

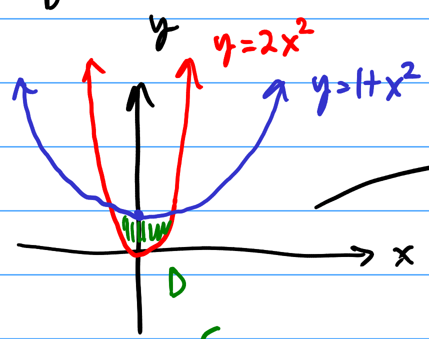
Type I : ↑ y-direction first

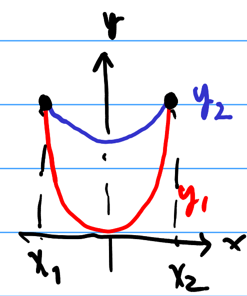
Type II : → x- direction first



$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Ex. $\iint_D (x+2y)\,dA$ D is bdd between $\begin{cases} y = 2x^2 \\ y = 1+x^2 \end{cases}$

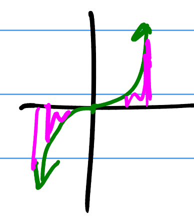




$$\int_{-1}^{1} \left[\int_{2x^2}^{1+x^2} (x+2y)\,dy \right] dx$$

$$2x^2 = 1 + x^2$$
$$x^2 = 1$$
$$x = \pm 1$$

$$= \int_{-1}^{1} \left[xy + y^2 \Big|_{y=2x^2}^{1+x^2} \right] dx$$



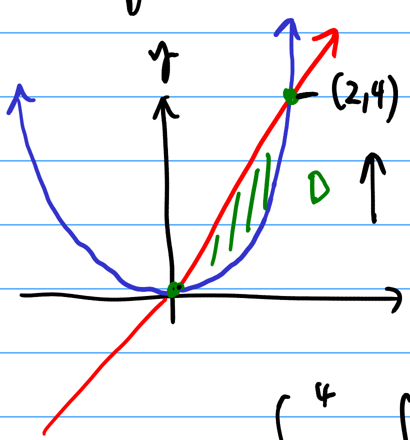
$$= \int_{-1}^{1} x(1+x^2) + (1+x^2)^2 - x(2x^2) - (2x^2)^2 \; dx$$

$$= \int_{-1}^{1} \cancel{x} + \cancel{x^3} + 1 + 2x^2 + x^4 - \cancel{2x^3} - 4x^4 \; dx$$

$$= 2\int_0^1 1 + 2x^2 - 3x^4 \, dx = 2\left(x + \frac{2}{3}x^3 - \frac{3}{5}x^5\right)\Big|_0^1$$

$$= 2\left(\frac{15+10-9}{15}\right) = \frac{32}{15}$$

Ex. $\iint_D x^2+y^2 \, dA$ D is bdd between $\begin{cases} \boxed{y = 2x} \to x = \frac{1}{2}y \\ y = x^2 \to x = \sqrt{y} \end{cases}$



D $\uparrow$ or $\rightarrow$ fun

$$x^2 = 2x$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$
$$x = 0 \quad x = 2$$

$$\int_0^4 \int_{x=\frac{1}{2}y}^{\sqrt{y}} x^2+y^2 \, dx \, dy$$

$$= \int_0^4 \left[\frac{1}{3}x^3 + xy^2\right]_{x=\frac{1}{2}y}^{\sqrt{y}} dy$$

$$= \int_0^4 \frac{1}{3}y^{3/2} + y^{5/2} - \frac{1}{3}\left(\frac{1}{2}y\right)^3 - \left(\frac{1}{2}y\right)y^2 \, dy$$

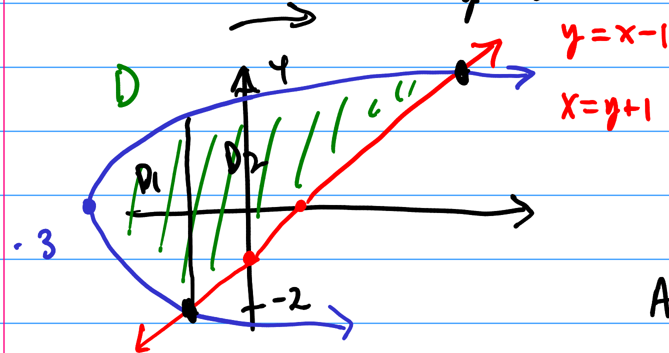
$$= \int_0^4 \frac{1}{3}y^{3/2} + y^{5/2} - \frac{1}{6}y^3 \, dy$$

$$= \frac{1}{3}\cdot\frac{2}{5}y^{5/2} + \frac{2}{7}y^{7/2} - \frac{1}{24}y^4 \Big|_0^4$$

$$= \frac{64}{15} + \frac{256}{7} - \frac{256}{24} = \boxed{\frac{1056}{35}}$$?

Ex. $\iint_D xy\,dA$ $\qquad y = x-1$

$y^2 = 2x+6 \rightarrow \quad x = \frac{1}{2}y^2 - 3$



As Type I: $\iint_{D_1} \bullet\, dA + \iint_{D_2} \bullet\, dA$

As Type II: $\frac{1}{2}y^2 - 3 \rightarrow y+1$

$$\boxed{\iint_D xy\,dA = \int_{\boxed{-2}}^{\boxed{4}} \int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy}$$

FTIS

$y+1 = \frac{1}{2}y^2 - 3$

$2y + 2 = y^2 - 6$

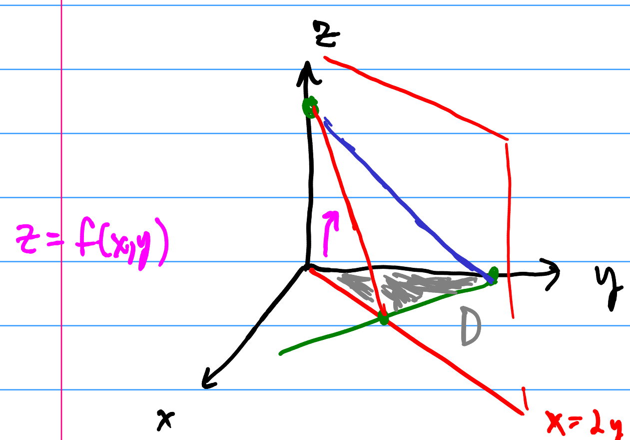
$y^2 - 2y - 8 = 0$

$(y+2)(y-4) = 0$

$y = -2 \qquad y = 4$

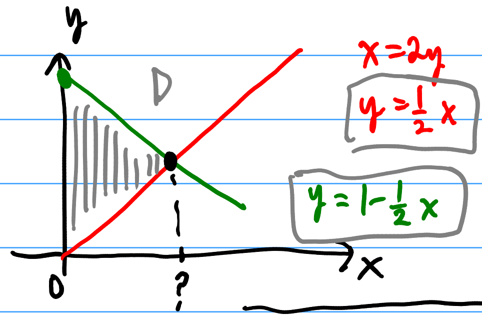
Ex. Set up a double integral representing the volume of the tetrahedron bdd by $\begin{cases} x+2y+z=2 \\ x=2y \\ x=0 \\ z=0 \end{cases}$ $\leftarrow$ $z=0$: $x+2y=2$, $y = -\frac{1}{2}x+1$

$(x,y)=(0,0)$: $z=2$



$V = \iint_D 2-x-2y\,dA$

Domain:



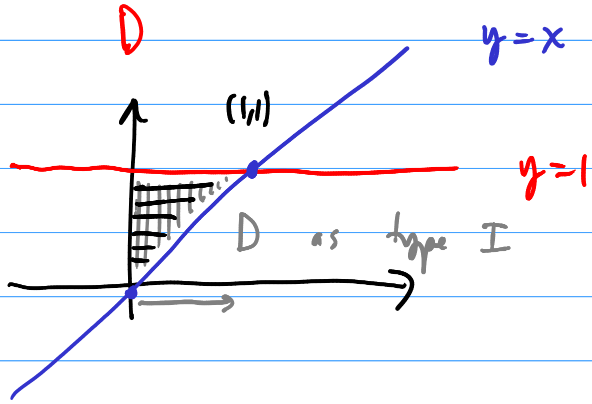
or $x = -2y + 2$

$$\frac{1}{2}x = 1 - \frac{1}{2}x$$
$$x = 1$$

$$V = \boxed{\int_0^1 \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} 2 - x - 2y \; dy \; dx} \quad \text{FTIS}.$$

Ex. $\underbrace{\boxed{\int_0^1} \boxed{\int_x^1}}_{D} \sin(y^2) \boxed{dy} \, dx$

$y = 1$
$y = x$ ↑



$x: 0 \to 1$

as type II:

$x = 0 \to x = y$
$y: 0 \to 1$

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \int_0^1 \underbrace{\int_0^y \sin(y^2)\, dx}\, dy$$

$$= \int_0^1 \left[x \sin(y^2) \Big|_{x=0}^{x=y} \right] dy$$

$$= \frac{1}{2} \int_0^1 2y \sin(y^2)\, dy$$

$$= -\frac{1}{2}\cos(y^2) \Big|_0^1 = -\frac{1}{2}\cos(1) + \frac{1}{2}\cos(0)$$

$$= \boxed{\frac{1}{2} - \frac{1}{2}\cos(1)}$$