M344 — 18 Oct '18

Ch 15 – Multiple Integrals

§15.9 – Change of Variables

Calc I: Ex. $\int x(\underbrace{x^2+3}_{})^9\,dx = \frac{1}{2}\int u^9\,du$

$u = x^2+3$

$\frac{1}{du/dx} = \frac{dx}{du}$

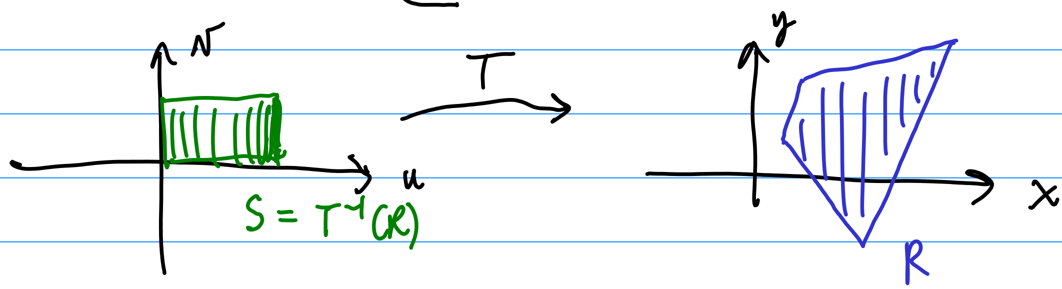
$du = 2x\,dx \rightarrow \underbrace{dx}_{} = \underbrace{\frac{1}{2x}\,du}_{}$

so, $\int f(x)\,dx = \int f(x(u))\,\underbrace{\frac{dx}{du}}_{}\,du$

here regard $x = x(u)$ as a function.

In 2D, (x,y) rectangular coordinates.

T a transformation: $T(u,v) = (x,y)$



Want: $\iint_R f(x,y)\,\underbrace{dA_{(x,y)}}_{} = \iint_S f(u,v)\,\underbrace{dA_{(u,v)}}_{}$

Question: How is $dA_{(u,v)}$ related to $dA_{(x,y)}$?

Defn. Let T be a one-to-one transformation $T = \begin{cases} x(u,v) \\ y(u,v) \end{cases}$

Then the Jacobian of T is $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix}$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

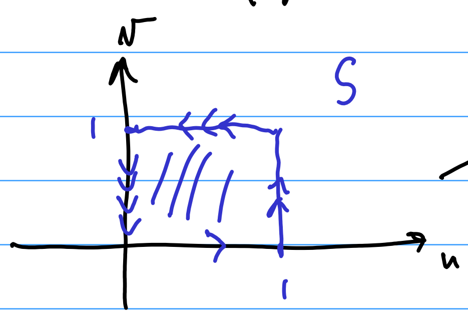
This is a function of u, v.

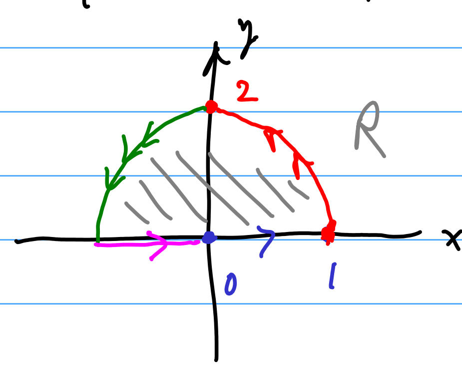
$$dA = dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

So,

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Ex. $T: \begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$ Find the image in the xy-plane of $S = \{(u,v) : 0 \le u \le 1,\ 0 \le v \le 1\}$





$\rightarrow$: $v = 0$. $u: 0 \to 1$

$x(\rightarrow) = u^2 \quad : 0 \to 1$

$y(\rightarrow) = 0$ b.c $v = 0$

$\uparrow$: $u = 1$, $v: 0 \to 1$

$x(\uparrow) = u^2 - v^2 = 1 - v^2 \quad : 1 \to 0$

$y(\uparrow) = 2uv = 2v \quad : 0 \to 2$

$y = 2v \rightarrow v = y/2$

$x = 1 - v^2 = 1 - (y/2)^2$

$x = 1 - \frac{1}{4}y^2$

Q. Are all transformations "acceptable"?

Ex. $T \begin{cases} x = \dfrac{u}{u+v} \\ y = \dfrac{u}{u-v} \end{cases}$ Compute the Jacobian.

$$\frac{\partial x}{\partial u} = \frac{(u+v)\cdot 1 - u}{(u+v)^2} = \frac{v}{(u+v)^2}$$

$$\frac{\partial x}{\partial v} = \frac{-u}{(u+v)^2}$$

$$\frac{\partial y}{\partial u} = \frac{(u-v)-u}{(u-v)^2} = \frac{-v}{(u-v)^2}$$

$$\frac{\partial y}{\partial v} = \frac{+u}{(u-v)^2}$$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$J = \begin{vmatrix} \frac{v}{(u+v)^2} & \frac{-v}{(u-v)^2} \\ \frac{-u}{(u+v)^2} & \frac{u}{(u-v)^2} \end{vmatrix}$$

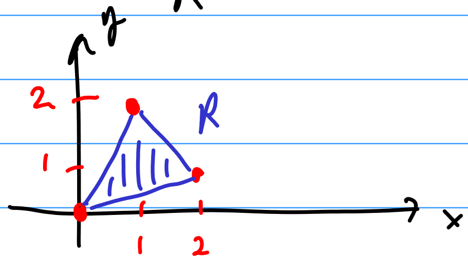
$$= \frac{vu - uv}{(u+v)^2(u-v)^2} = 0$$

$$dx\,dy = |J|\,du\,dv = \underline{0}\,du\,dv$$ NOT acceptable.

This T collapsed a dimension, zeroing the area.

So, $J \neq 0$ for acceptable transformations.

Ex. $\iint_R (x-3y)\,dA$ R is the triangle w/ vertices $(0,0)$, $(2,1)$, $(1,2)$.



$$T = \boxed{\begin{cases} x = 2u + v \\ y = u + 2v \end{cases}}$$

1. Find S. $T(S)=R$, $S=T^{-1}(R)$

Solve for u, v both as functions of (x,y)

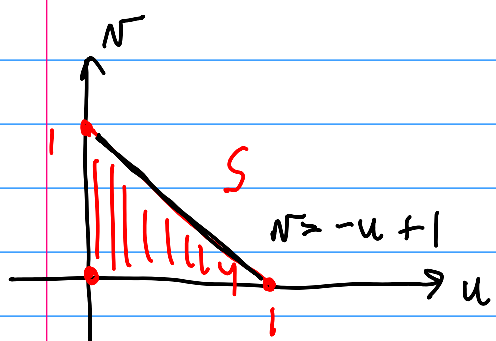
$$x = 2u + v \qquad\qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$
$$y = u + 2v$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$u = \frac{|A_1|}{|A|} \qquad v = \frac{|A_2|}{|A|}$$

$$A_1 = \begin{pmatrix} x & 1 \\ y & 2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 2 & x \\ 1 & y \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$



$$|A| = 3 \qquad |A_1| = 2x - y \qquad |A_2| = 2y - x$$

$$\boxed{u = \frac{2}{3}x - \frac{1}{3}y \ , \quad v = -\frac{1}{3}x + \frac{2}{3}y}$$

(x,y)	(u,v)
$(0,0)$	$(0,0)$
$(1,2)$	$(0,1)$
$(2,1)$	$(1,0)$

2. Find the Jacobian:

$$\begin{cases} x = 2u + v \\ y = u + 2v \end{cases} \qquad |J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$$dA = 3\,du\,dv = 3\,dv\,du$$

3. Change the integrand: $x - 3y = \overbrace{(2u+v)}^{x} - 3\overbrace{(u+2v)}^{y}$

$$= -u - 5v$$

4. Integrate!

$$\iint_R x-3y \, dA = \iint_S (-u-5v)\,3\,dA$$

$$= \int_0^1 \int_0^{1-u} (-u-5v)\,3 \, dv \, du$$

$$= 3\int_0^1 \int_0^{1-u} -u-5v \, dv\, du$$

$$= \text{FTIS} = \boxed{-3}$$

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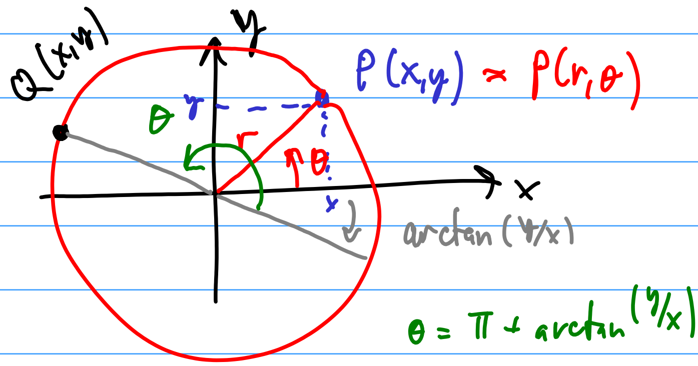
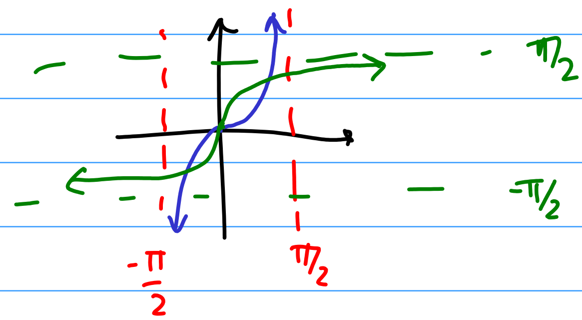
§15.3 – Polar Coordinates



$$\boxed{T = \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}}$$

$$T^{-1} = \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan(y/x) \end{cases} \leftarrow \text{adjust for quad.}$$



Ex. Find the Jacobian of the polar transformation

$$T = \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

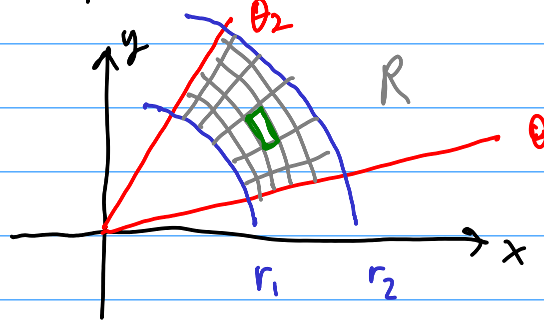
$$|J| = \begin{vmatrix} \partial_r x & \partial_r y \\ \partial_\theta x & \partial_\theta y \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

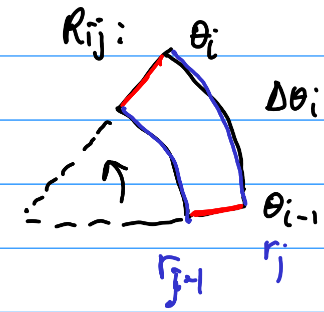
$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r$$

therefore, $\boxed{dA = dx\,dy = r\,dr\,d\theta}$ (*)

Geometrically, a <u>polar rectangle</u> is $[r_1, r_2] \times [\theta_1, \theta_2]$

Area of sector
$A = \frac{1}{2} r^2 \theta$





$$\Delta A_{ij} = \frac{1}{2} r_j^2 \Delta\theta_i - \frac{1}{2} r_{j-1}^2 \Delta\theta_i$$

$$= \frac{1}{2}\left(r_j^2 - r_{j-1}^2\right)\Delta\theta_i$$

$$= \underbrace{\frac{1}{2}(r_j + r_{j-1})}\,\underbrace{(r_j - r_{j-1})}_{\Delta r_j}\,\Delta\theta_i$$

In the limit we get:

$$\lim_{i,j\to\infty} \frac{1}{2}(r+r)\,\Delta r\,\Delta\theta = \boxed{r\,dr\,d\theta = dA}$$