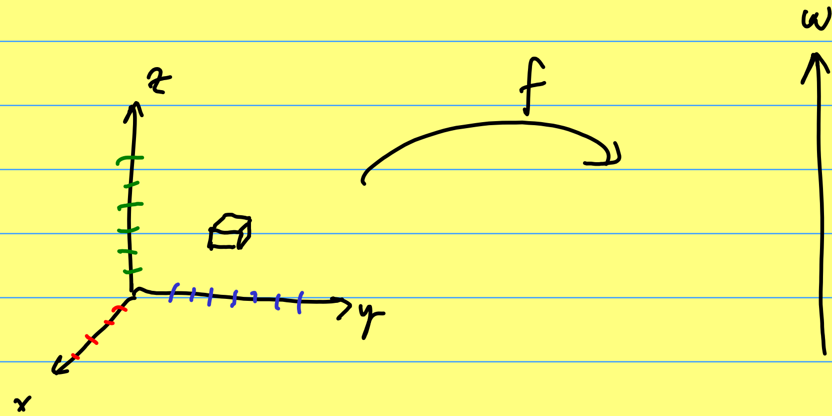
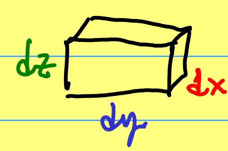
M344 23 oct '18

§15.7,8,9 – Triple Integrals

$f = f(x,y,z)$



$$\iiint_E f(x,y,z)\, dV$$



In (x,y,z) coord's $dV = dx\,dy\,dz$

Ex. $\displaystyle\iiint_B xyz^2\, dV \qquad B = [0,1]\times[-1,2]\times[0,3]$

$$= \int_0^1 \int_{-1}^2 \left[\int_0^3 xyz^2\, dz\right] dy\, dx$$

$$= \int_0^1 x\,dx \cdot \int_{-1}^2 y\,dy \cdot \int_0^3 z^2\,dz$$

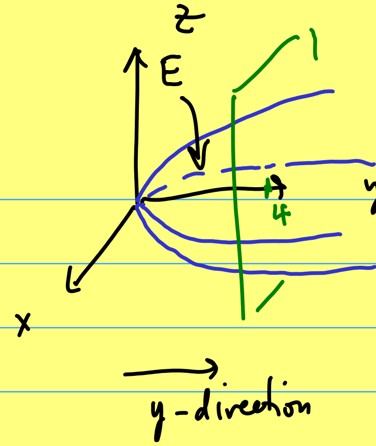
$$= \left(\tfrac{1}{2}x^2\Big|_0^1\right)\cdot\left(\tfrac{1}{2}y^2\Big|_{-1}^2\right)\cdot\left(\tfrac{1}{3}z^3\Big|_0^3\right) = \tfrac{1}{2}\cdot\left(2-\tfrac{1}{2}\right)(9) = \frac{27}{4}$$

Ex. $\displaystyle\iiint_E \sqrt{x^2+z^2}\, dV \qquad E$ is bdd between

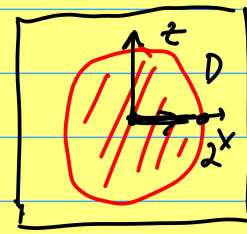
$y_1 = x^2+z^2$

$y_2 = 4$

Sketch the region E:



$x^2+z^2 \le 4$



$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_D \left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy \right] dA$$

$$= \iint_D \Big(4-\underbrace{(x^2+z^2)}_{r^2}\Big)\underbrace{\sqrt{x^2+z^2}}_{r}\,dA$$

Polar!:

$x = r\cos\theta$

$z = r\sin\theta$

$\boxed{dA = r\,dr\,d\theta}$

$r: 0 \to 2$

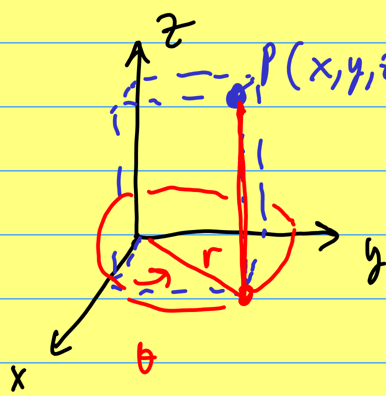
$\theta: 0 \to 2\pi$

$$= \int_0^{2\pi}\int_0^{2} (4-r^2)\,r\; r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^2 4r^2 - r^4\,dr$$

$$= 2\pi \cdot \left(\frac{4}{3}r^3 - \frac{1}{5}r^5 \Big|_0^2 \right) = 2\pi\left(2^5\left(\frac{1}{3}-\frac{1}{5}\right)\right) = 64\pi \cdot \frac{2}{15} = \frac{128\pi}{15}$$

Cylindrical Coordinates:



$P(x,y,z) = (r,\theta,z)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

TBD: Compute Jacobian, $J = \dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{pmatrix} \partial x/\partial r & \partial y/\partial r & \partial z/\partial r \\ \partial x/\partial \theta & \partial y/\partial \theta & \partial z/\partial \theta \\ \partial x/\partial z & \partial y/\partial z & \partial z/\partial z \end{pmatrix}$

$$J = \begin{array}{ccc} + & - & + \\ \end{array}$$
$$J = \begin{pmatrix} \cos\theta & \cancel{\sin\theta} & \cancel{0} \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|J| = \cos\theta\left(r\cos\theta - 0\right) - \sin\theta\left(-r\sin\theta - 0\right) + 0$$

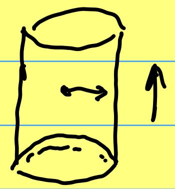
$$= r\cos^2\theta + r\sin^2\theta = r$$

So, $\boxed{dV = dx\,dy\,dz = \boxed{r}\,dr\,d\theta\,dz}$

<u>Ex.</u> Solid bdd between: $x^2+y^2=1$ (inside)

$z=4$ (upper bd)

$z=1-x^2-y^2$ (lower bd) Find <u>volume</u>.



cylindrical!

$1-(x^2+y^2) = 1-r^2$

$r=1$, $r: 0 \to 1$ $z: 1-r^2 \to 4$

$\theta: 0 \to 2\pi$

$$V = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 1\cdot r\,dz\,dr\,d\theta$$

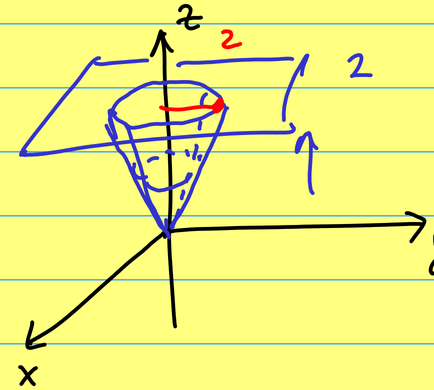
$= \text{FTIS} =$ the volume.

<u>Idea</u>: $\text{Area}(D) = \iint_D 1\,dA$

$\text{Vol}(E) = \iiint_E 1\,dV$

$s = \int_a^b ds$

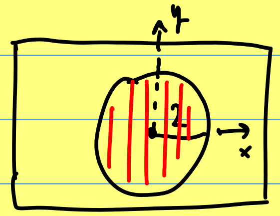
Ex. $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} \overbrace{\boxed{(x^2+y^2)}}^{r^2}\,\overbrace{dz\,dy\,dx}^{r\,dz\,dr\,d\theta}$



$z: \sqrt{x^2+y^2} \to 2$

$z = \sqrt{x^2+y^2}$ Cone!

$z = r$



$x^2+y^2=4 \to$ $y = \sqrt{4-x^2}$

$y = -\sqrt{4-x^2}$

then $x: -2 \to 2$!

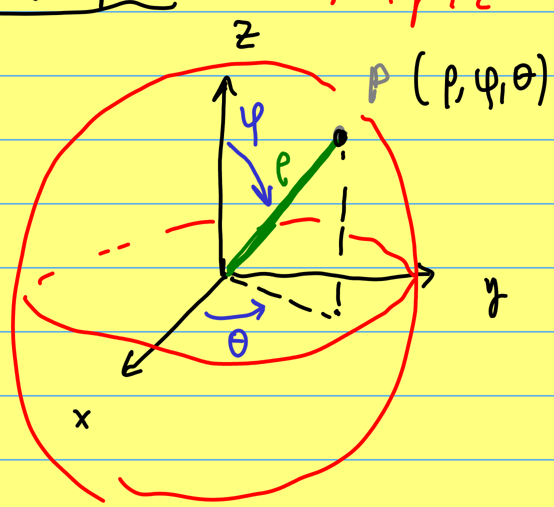
the whole circle!

The integral in cylindrical coord's equals

$$\int_0^{2\pi}\int_0^{2}\int_r^{2} r^2 \, r \, dz\,dr\,d\theta = \text{FTIS} =$$

Spherical Coordinates $\quad x^2+y^2+z^2=\rho^2$

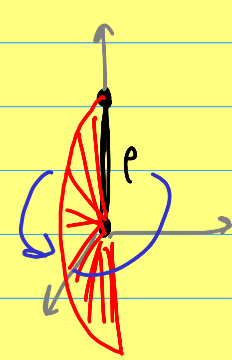
$\phi = \varphi = \varphi$



$\rho \geq 0$

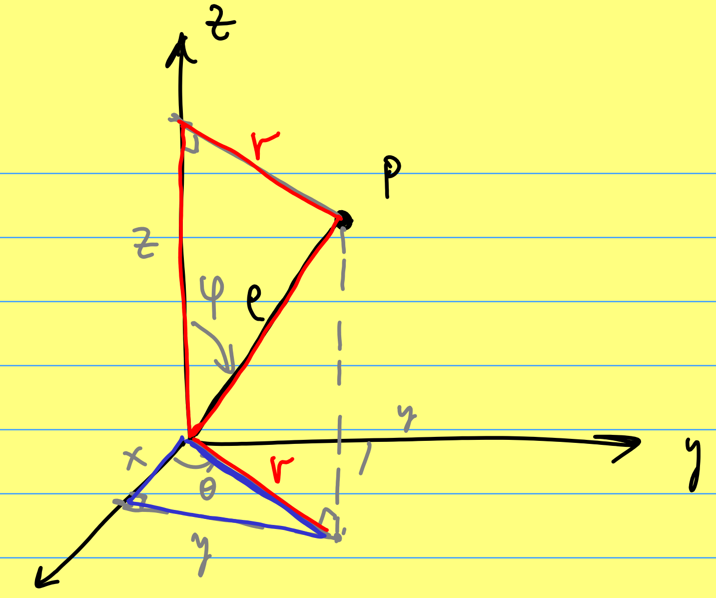
$0 \leq \varphi \leq \pi$

$0 \leq \theta < 2\pi$



Need x, y, z as functions of ρ, φ, θ.

Zoom in:



$\rightarrow z = \rho \cos\varphi$

$r = \rho \sin\varphi \leftarrow$

$x = r\cos\theta = \rho\sin\varphi\cos\theta$

$y = r\sin\theta = \rho\sin\varphi\sin\theta$

$$\boxed{T: \begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}}$$

Now, compute the Jacobian:

$$J = \begin{pmatrix} \partial_\rho x & \partial_\rho y & \partial_\rho z \\ \partial_\varphi x & \partial_\varphi y & \partial_\varphi z \\ \partial_\theta x & \partial_\theta y & \partial_\theta z \end{pmatrix} = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{pmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\sin\theta & -\rho\sin\varphi \\ -\rho\sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & +0 \end{pmatrix} \end{array}$$

$$|J| = \cos\varphi\left(\rho^2\cos\varphi\sin\varphi\cos^2\theta + \rho^2\cos\varphi\sin\varphi\sin^2\theta\right) +$$

$$+\rho\sin\varphi\left(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta\right)$$

$$= \cos\varphi\left(\rho^2\cos\varphi\sin\varphi\right) + \rho\sin\varphi\left(\rho\sin^2\varphi\right)$$

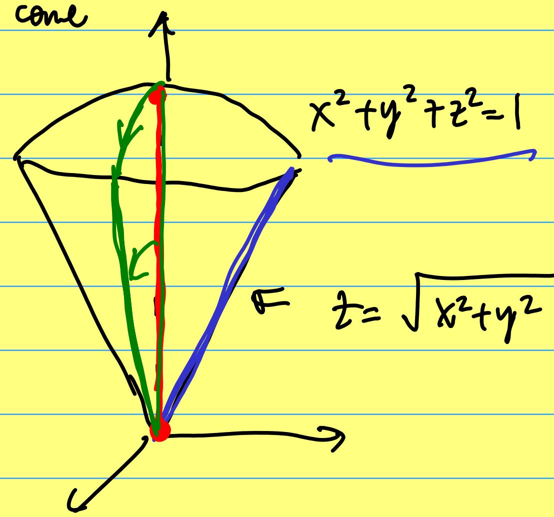
$$= \rho^2\sin\varphi\left(\cos^2\varphi + \sin^2\varphi\right)$$

$$= \underline{\rho^2\sin\varphi} \quad \blacksquare$$

$\rho^2\sin\varphi = \rho\cdot\rho\sin\varphi = \rho\cdot r$ ✓

$$\boxed{dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta} \quad (*)$$

Find volume of snow cone



$\rho: 0 \to 1$

$\varphi: 0 \to \pi/4$

$\theta: 0 \to 2\pi$

$$V = \iiint_E 1 \, dV = \underbrace{\int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta}_{\text{FTIS.}}$$