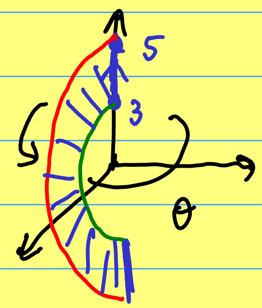
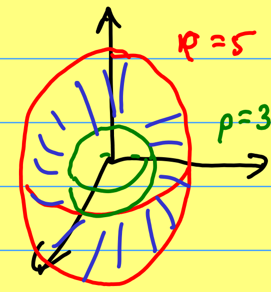
Ex. Find the volume enclosed between the spheres

$$x^2+y^2+z^2=25$$
$$x^2+y^2+z^2=9$$




$\rho: 3 \to 5$
$\varphi: 0 \to \pi$
$\theta: 0 \to 2\pi$

$$V = \iiint_E 1 \, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_3^{5} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \sin\varphi \, d\varphi \cdot \int_3^5 \rho^2 \, d\rho$$

$$= 2\pi \left[ -\cos\varphi \Big|_0^{\pi} \right] \cdot \left[ \frac{1}{3}\rho^3 \Big|_3^5 \right]$$

$$= 2\pi \left( -(-1) + (1) \right) \frac{1}{3} \; \rho^3 \Big|_3^5$$

$$= \frac{4}{3}\pi \rho^3 \Big|_3^5 = \frac{4}{3}\pi (125-27) = \frac{392}{3}\pi$$

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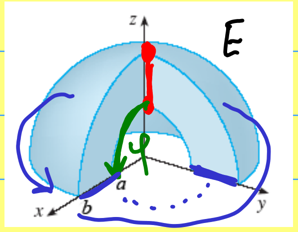


Write a triple integral representing the volume

$$V = \iiint_E 1 \, dV = \int_{\pi/2}^{2\pi}\int_0^{\pi/2}\int_a^b \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

**12.** Question Details — SCalc8 15.8.023. [3351071]

Use spherical coordinates.

Evaluate $\iiint_E (x^2+y^2)\,dV$, where $E$ lies between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$.

$2^2 \qquad 3^2$

$f(x,y,z) = x^2+y^2$ Needs to be spherical!

Recall:

$$\left\{\begin{array}{l} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ \boxed{z = \rho\cos\varphi} \end{array}\right.$$

$\boxed{r = \rho\sin\varphi}$ in the xy-plane (polar)

and

$r^2 = x^2+y^2$

$f(x,y,z) = r^2 = (\rho\sin\varphi)^2 = \rho^2\sin^2\varphi$ $\quad dV$

So,
$$\int_0^{2\pi}\int_0^{\pi}\int_2^{3} \rho^2\sin^2\varphi \;\; \rho^2\sin\varphi \; d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\pi}\sin^3\varphi\, d\varphi \cdot \int_2^3 \rho^4\, d\rho = \text{FTIS}$$

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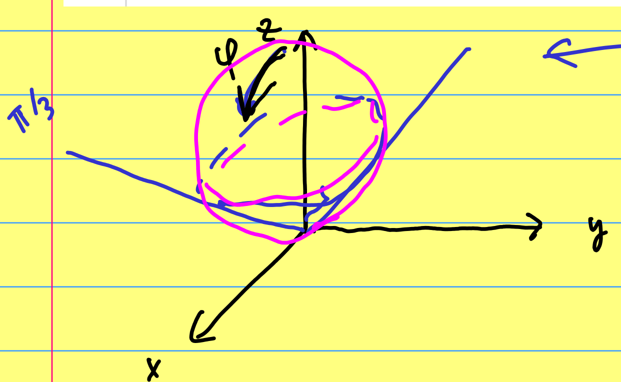
**13.** Question Details

Use spherical coordinates.

(a) Find the volume of the solid that lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 12\cos(\phi)$.




$$\left\{\begin{array}{l} \rho: 0 \to 12\cos\varphi \\ \varphi: 0 \to \pi/3 \\ \theta: 0 \to 2\pi \end{array}\right.$$

$\rho = 12\cos\varphi$

$\rho(\pi/3) = 12\cos(\pi/3) = 12\cdot\frac{1}{2} = 6$

$\cancel{\rho^2 = 12^2\cos^2\varphi \;\rightarrow\; x^2+y^2+z^2 = 12^2\cos^2\varphi}$

$\rho = 12\cos\varphi = 12\left(z/\rho\right)$

$\rho^2 = 12z$

$x^2 + y^2 + z^2 = 12z$

$x^2 + y^2 + (z^2 - 12z + 36) = 0 + 36$

$x^2 + y^2 + (z-6)^2 = 6^2$

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_0^{12\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\pi/3} \left[ \sin\varphi \int_0^{12\cos\varphi} \rho^2 \, d\rho \right] d\varphi$$

$$= 2\pi \cdot \int_0^{\pi/3} \sin\varphi \left[ \frac{1}{3} 12^3 \cos^3\varphi \right] d\varphi$$

$$= -\frac{2^7 \cdot 3^3}{3} \pi \int_0^{\pi/3} -\sin\varphi \cos^3\varphi \, d\varphi$$

$u = \cos\varphi$ $\qquad u(\pi/3) = 1/2$

$du = -\sin\varphi \, d\varphi$ $\qquad u(0) = 1$

$$= 2^7 \cdot 3^2 \pi \int_{1/2}^{1} u^3 du = \frac{2^7 \cdot 3^2 \pi}{2^2} \left( 1^4 - \left(\frac{1}{2^4}\right) \right)$$

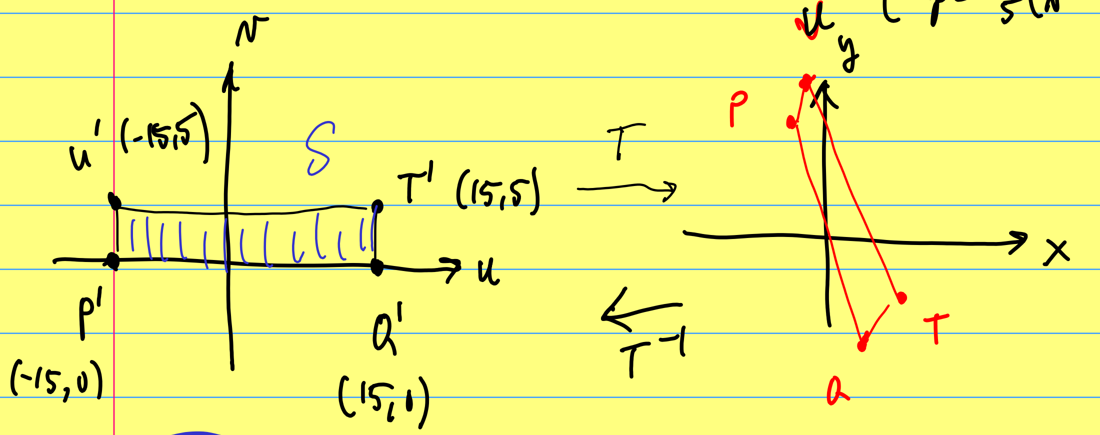
$$= \frac{2^{\cancel{7}} \cdot 3^2 \pi}{2^{\cancel{6}}} \left( \frac{16-1}{\phantom{1}} \right)$$

$$= \boxed{270\pi}$$

$\iint_R (5x+10y)\,dA$

$R = \text{//-ogram}$ $\begin{cases} P(-3,12) & T(4,-1) \\ Q(3,-12) & U(-2,13) \end{cases}$

$\begin{cases} x = \frac{1}{5}(u+v) \\ y = \frac{1}{5}(v-4u) \end{cases}$ ← solve for u, v




$\boxed{\begin{array}{l} u: -15 \to 15 \\ v: 0 \to 5 \end{array}}$

$\begin{cases} 5x = u+v \\ 5y = v-4u \end{cases}$

$A = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$ $\quad A_1 = \begin{pmatrix} 5x & 1 \\ 5y & 1 \end{pmatrix}$ $\quad A_2 = \begin{pmatrix} 1 & 5x \\ -4 & 5y \end{pmatrix}$

$|A| = 5$ $\quad |A_1| = 5x - 5y$ $\quad |A_2| = 5y + 5\cdot 4x$

$$u = \frac{|A_1|}{|A|} = x - y$$

$$v = \frac{|A_2|}{|A|} = y + 4x = 4x + y$$

$$\iint_R (5x+10y)\,dA = \int_0^5 \int_{-15}^{15} \left(5\cdot\frac{1}{5}(u+v) + 10\cdot\frac{1}{5}(v-4u)\right) \underbrace{|J|}\, du\, dv$$

$$J = \begin{pmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{pmatrix} = \begin{pmatrix} 1/5 & -4/5 \\ 1/5 & 1/5 \end{pmatrix} \qquad |J| = \frac{1}{25} + \frac{4}{25} = 1/5$$

$$= \frac{1}{5}\int_0^5 \int_{-15}^{15} 3v - 7u \; du \, dv = \ldots$$

$$\frac{1}{5}\int_0^5 90v \, dv = 225 \, ?$$