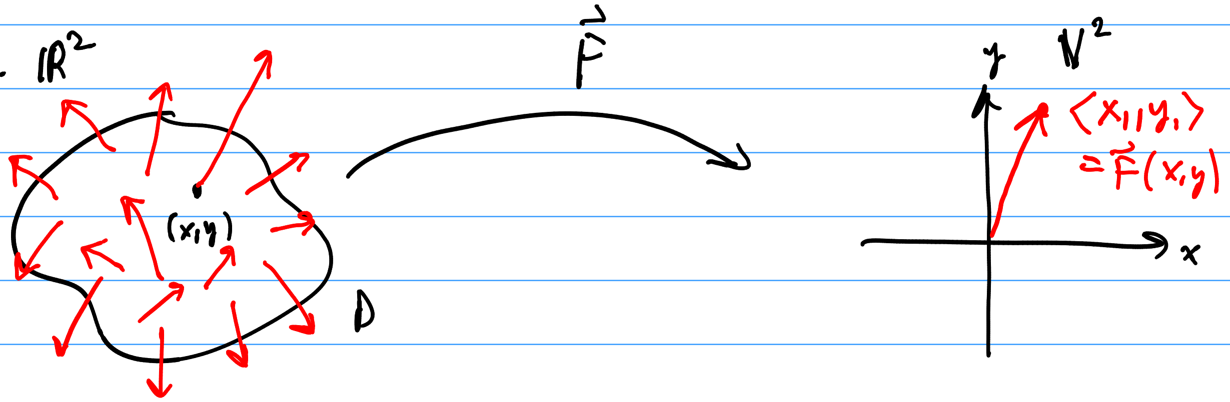
M344 30 oct 18

Ch16 Vector Calculus

§16.1 – Vector Fields

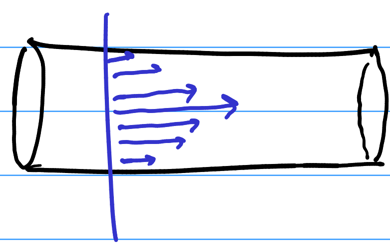
Let $D \subseteq \mathbb{R}^n$ a domain (or region). A vector field on D is a function $\vec{F}: D \to V^n$ = vectors in $\mathbb{R}^n$.

Ex. $\mathbb{R}^2$



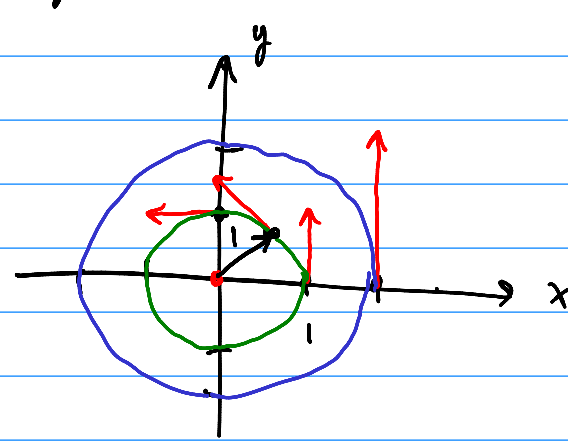
Ex. Couette Flow "Pipe Flow"

velocity of water flowing in straight pipe



Ex. $\vec{F}(x,y) = \langle -y, x \rangle$ for all (x,y) in $\mathbb{R}^2$

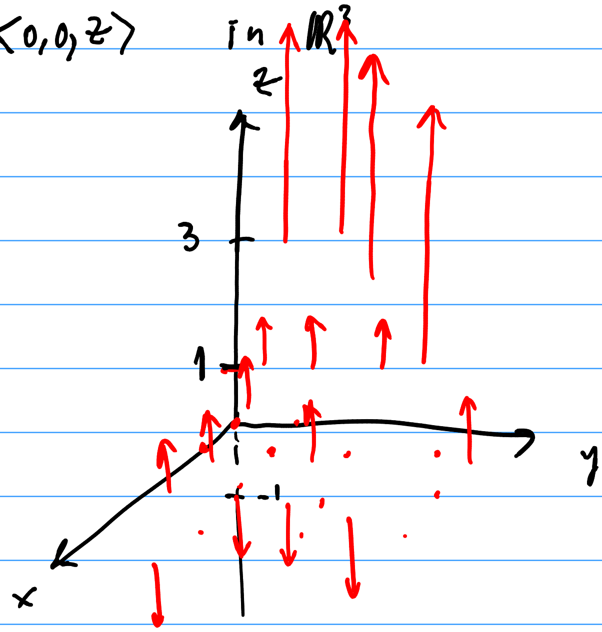
$\vec{r} = (x,y)$	$\vec{F}(x,y)$
$(0,0)$	$\langle 0,0 \rangle$
$(1,0)$	$\langle 0,1 \rangle$
$(0,1)$	$\langle -1,0 \rangle$
$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$ ←
$(2,0)$	$\langle 0,2 \rangle$



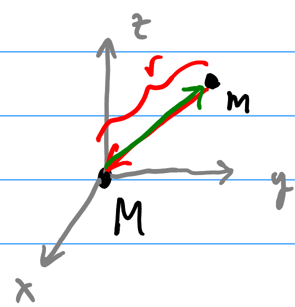
Check: $\vec{x} \perp \vec{F}(\vec{x})$

$$\vec{x}\cdot\vec{F} = \langle x,y\rangle\cdot\langle -y,x\rangle = -xy + xy = 0 \checkmark$$

Ex. $\vec{F}(x,y,z) = \langle 0,0,z\rangle$ in $\mathbb{R}^3$



Ex. Newton's Law of Gravity: $\|\vec{F}(r)\| = \dfrac{mMG}{r^2}$ Inverse square law



$m=$ mass of free object
$M=$ mass of fixed object
$G=$ gravitational constant

$\vec{x} = \langle x,y,z\rangle$ is position of m

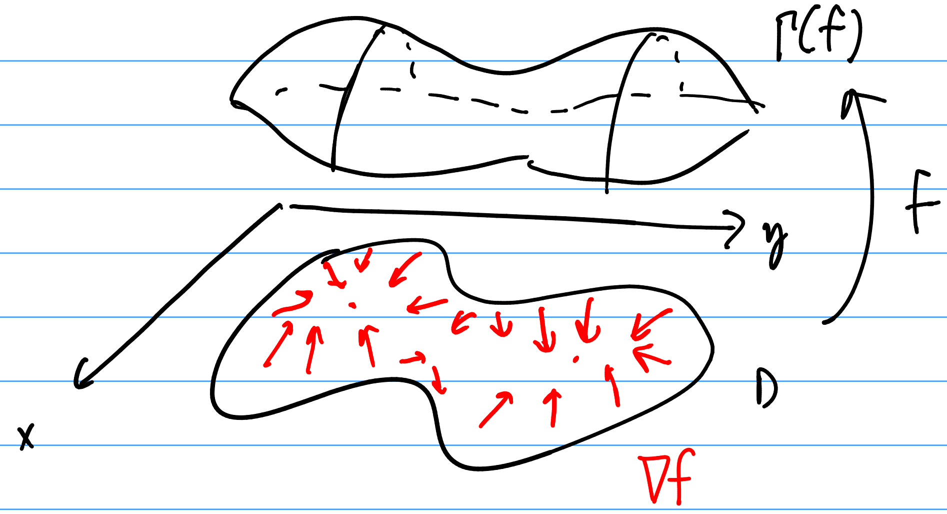
then $r = \|\vec{x}\| = \sqrt{x^2+y^2+z^2}$

so, $\vec{F}(\vec{x}) = \dfrac{-\vec{x}}{\|\vec{x}\|}\|\vec{F}(\vec{x})\| = \boxed{-\vec{x}\dfrac{mMG}{\|\vec{x}\|^3}} = -\langle x,y,z\rangle\dfrac{mMG}{\|\vec{x}\|^3}$

or

$$\boxed{\vec{F}(x,y,z) = \left\langle \frac{-x\,mMG}{\sqrt{x^2+y^2+z^2}^{\,3}}, \frac{-y\,mMG}{\sqrt{x^2+y^2+z^2}^{\,3}}, \frac{-z\,mMG}{\sqrt{x^2+y^2+z^2}^{\,3}} \right\rangle}$$

Ex. Let f be a function of 2 or more variables. The gradient of f is a natural vector field on the domain of f.



Defn. Let $\vec{F}$ be a vector field on $D \subseteq \mathbb{R}^n$. $\vec{F}$ is said to be conservative if and only if there exists a function $f: D \to \mathbb{R}$ such that

$$\vec{F}(x,y) = \nabla f(x,y) \quad \text{for } (x,y) \text{ in } D.$$

f is a potential function for $\vec{F}$.

→ So potential functions play the same role as antiderivatives, but for vector fields.

Ex. The gravitational potential function is

$$f(\vec{x}) = \frac{mMG}{\|\vec{x}\|^2} \quad \leftarrow \text{LIE}$$

or $$f(x,y,z) = \frac{mMG}{x^2+y^2+z^2}$$

$$\frac{\partial f}{\partial x} = -\frac{2x\,mMG}{(x^2+y^2+z^2)^2} = \frac{-2x\,mMG}{\|\vec{x}\|^4}$$

NO. ↑

yes ↓

Instead, $f(x,y,z) = \dfrac{mMG}{\sqrt{x^2+y^2+z^2}}$ $\qquad \dfrac{\partial f}{\partial x} = \dfrac{-1}{\cancel{2}} \cdot \dfrac{\cancel{2}x\, mMG}{\sqrt{x^2+y^2+z^2}^{\,3}}$ ✓

$$f(\vec{x}) = \frac{mMG}{\|\vec{x}\|} \quad \text{and} \quad \vec{F} = \nabla f.$$

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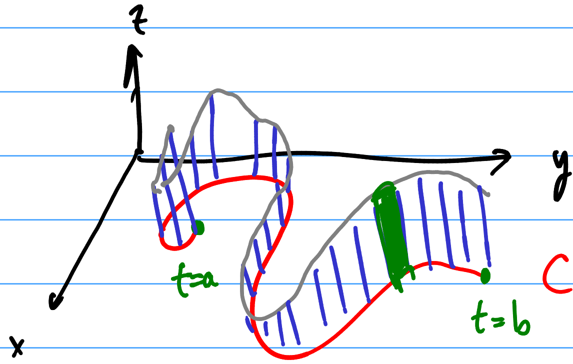
§16.2 Path Integrals

Suppose C is a smooth curve in $\mathbb{R}^2$, parametrized by $\vec{r} = \vec{r}(t)$.
And let f be function defined on some domain containing C.

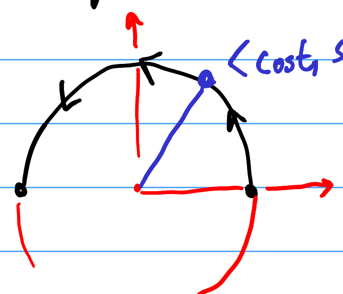


$C : \vec{r}(t) = \langle x(t), y(t) \rangle$

want: Area under f along C

$$\boxed{\int_C f(x,y)\, ds = \int_a^b f(\vec{r}(t))\, \|\dot{\vec{r}}\|\, dt} \quad (*)$$

Ex. $\displaystyle\int_C (2+x^2y)\, ds$ $\qquad C =$ upper half of unit circle $x^2+y^2=1$ counterclockwise.



$ds = \|\dot{\vec{r}}\|\, dt$

$\dot{\vec{r}} = \langle -\sin t, \cos t \rangle$ and $\|\dot{\vec{r}}\| = 1$

$$= \int_0^{\pi} \left(2 \overset{dt}{+} \int_0^{\pi} \underline{\cos^2} t \sin t \right) dt$$

$u = \cos t$

$du = -\sin t \, dt$

$u(\pi) = -1$

$u(0) = 1$

$$= \int_0^{\pi} 2 \, dt + 2 \int^{1} u^2 \, du$$

$$\boxed{2\pi + 2/3}$$