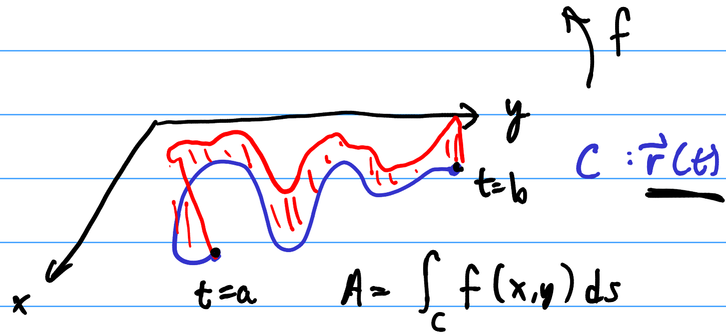
§16.2 Path Integrals



$$A = \int_C f(x,y)\,ds$$

$$= \int_a^b f(\vec{r}(t))\,\|\dot{\vec{r}}(t)\|\,dt$$

Partial Path Integrals :

$\vec{r}(t) = \langle x(t), y(t) \rangle$

$\rightarrow d\vec{r} = \langle dx, dy \rangle$

$\frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \langle \dot{x}, \dot{y} \rangle$

$d\vec{r} = \langle \dot{x}\,dt, \dot{y}\,dt \rangle$

$$\int_C f(x,y)\,dx = \int_a^b f(\vec{r}(t))\,\dot{x}(t)\,dt$$

and

$$\int_C f(x)\,dx = \int_a^b f(\vec{r}(t))\,\dot{y}(t)\,dt$$

Given a vector field, $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$,

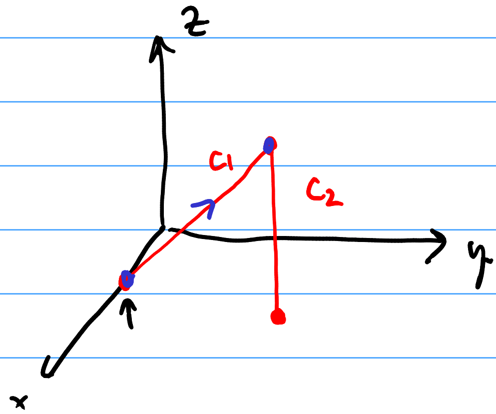
$$\rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_C \langle P, Q \rangle \cdot \langle dx, dy \rangle = \int_C \left(P\,dx + Q\,dy \right)$$

$$= \int_C P\,dx + \int_C Q\,dy$$

$$= \int_C P\,dx + Q\,dy$$

Ex. $\vec{F}(x,y,z) = \langle y, z, x \rangle$

$C: \begin{cases} (2,0,0) \to (3,4,5) & \text{line segment} \\ (3,4,5) \to (3,4,0) \end{cases}$



$$\int_C \vec{F} \cdot d\vec{r} = \underline{\int_{C_1} \vec{F} \cdot d\vec{r}} + \underline{\int_{C_2} \vec{F} \cdot d\vec{r}}$$

$d\vec{r} = \langle dx, dy, dz \rangle$

$d\vec{r} = \langle \dot{x}\,dt, \dot{y}\,dt, \dot{z}\,dt \rangle$

C_1 : $\vec{r}_1(t) = (1-t)\langle 2,0,0 \rangle + t\langle 3,4,5 \rangle \qquad 0 \le t \le 1$

$\vec{r}_1(t) = \langle 2+t, 4t, 5t \rangle$

C_2 : $\vec{r}_2(t) = (1-t)\langle 3,4,5 \rangle + t\langle 3,4,0 \rangle \qquad 0 \le t \le 1$

$\to \vec{r}_2(t) = \langle 3, 4, 5-5t \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 \langle 4t, 5t, 2+t \rangle \cdot \langle \cancel{dt}\,1, 4\cancel{dt}, 5\cancel{dt} \rangle\, dt$$

$$= \int_0^1 4t + 20t + 10 + 5t \, dt$$

$$= \int_0^1 29t + 10 \, dt = \frac{29}{2} + 10 = \frac{49}{2}$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 \langle \underline{4}, \underline{5-5t}, 3 \rangle \cdot \langle \underline{0dt}, \underline{0dt}, -5dt \rangle$$

$$= \int_0^1 3 \cdot (-5) \, dt = -15$$

$$\int_C \vec{F} \cdot d\vec{r} = \frac{49}{2} + (-15) = \frac{19}{2}$$

Ex. $\int_C \underbrace{y \sin z}\, ds$ $\quad C: \vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$

$$\dot{\vec{r}} = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\dot{\vec{r}}\| = \sqrt{\sin^2 t + \cos^2 t + 1^2} = \sqrt{2}$$

$$= \int_0^{2\pi} \sin t \sin t \; \sqrt{2} \; dt$$

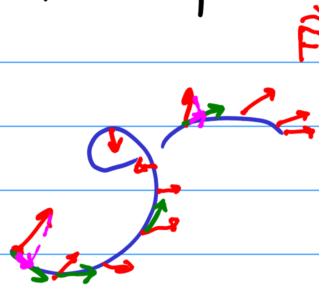
$$= \sqrt{2} \int_0^{2\pi} \sin^2 t \, dt = \frac{\sqrt{2}}{2} \int_0^{2\pi} 1 - \cos(2t) \, dt$$

$$= \frac{\sqrt{2}}{2} \left(2\pi - \underbrace{\cancel{\frac{1}{2} \sin(2t) \Big|_0^{2\pi}}}_{0} \right)$$

$$= \sqrt{2}\,\pi$$

Work. $\vec{F}(x,y,z)$ = force field

$\vec{r}$ = space curve thought of as the path of a particle.



$C: \vec{r}(t)$ $\qquad \vec{T}$ = unit tangent vector field

First.

Work $\propto$ projection of $\vec{F}$ onto $\vec{T}$ (magnitude)

$$\vec{p} = \text{proj}_{\vec{T}} \vec{F} = \frac{\vec{F} \cdot \vec{T}}{\cancel{\vec{T} \cdot \vec{T}}} \vec{T} = (\vec{F} \cdot \vec{T}) \vec{T}$$

$$|dW| = \|(\vec{F} \cdot \vec{T}) \vec{T}\| = \vec{F} \cdot \vec{T}$$

Total work done by $\vec{F}$ on the particle is

$$W = \int_C \vec{F} \cdot \vec{T} \, ds = \int_C \vec{F} \cdot \left(\frac{\dot{\vec{r}}}{\cancel{\|\dot{\vec{r}}\|}} \right) \cancel{\|\dot{\vec{r}}\|} \, dt$$

$$= \int_C \vec{F} \cdot \underbrace{\dot{\vec{r}} \, dt}_{d\vec{r}}$$

$$\boxed{W = \int_C \vec{F} \cdot d\vec{r}}$$

Ex. $\int_C \vec{F}\cdot d\vec{r}$, $\quad \vec{F}=\langle xy, yz, zx\rangle$, $\quad \vec{r}=\langle t, t^2, t^3\rangle$, $\quad 0\le t\le 1$

$$\vec{F}(\vec{r}(t)) = \langle t\cdot t^2,\ t^2\cdot t^3,\ t^3\cdot t\rangle = \langle t^3, t^5, t^4\rangle$$
$$d\vec{r} = \dot{\vec{r}}\,dt = \langle 1, 2t, 3t^2\rangle\,dt$$

$$W=\int_C \vec{F}\cdot d\vec{r} = \int_0^1 t^3+2t^6+3t^6\,dt = \frac{1}{4}+\frac{5}{7} = \frac{27}{28}$$

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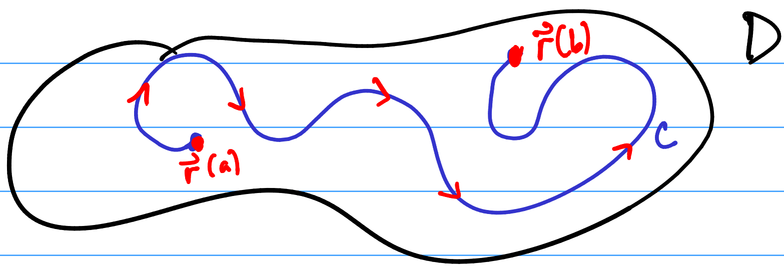
§16.3 – Fundamental Theorem of Path Integrals

Thm. Let $\vec{F}$ be a conservative vector field in a domain D. Let C be any path contained in D, parametrized by $\vec{r}(t)$, $a\le t\le b$. Then,

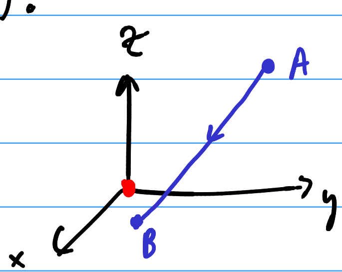
$$W = \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$



Ex. Gravitational v.f.

$$\vec{F}(x,y,z) = \frac{-mMG}{\|\vec{x}\|^3}\vec{x} \qquad f(\vec{x}) = \frac{mMG}{\|\vec{x}\|} \text{ is gravitational potential}$$

Find the work done by gravity to move a particle from $(3,4,12)$ to $(2,2,0)$.



$$W=\int_C \vec{F}\cdot d\vec{r} = f(B)-f(A)$$
$$= mMG\left(\frac{1}{\sqrt{2^2+2^2}} - \frac{1}{\sqrt{3^2+4^2+12^2}}\right)$$
$$= mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right) = \frac{mMG(13-2\sqrt{2})}{26\sqrt{2}}$$

Ex. Let $\vec{F}$ be conservative, so $\vec{F} = \nabla f$. The potential energy of an object acted on by $\vec{F}$ is defined to be $P(x,y,z) = -f(x,y,z)$.

So, $\vec{F} = -\nabla P$

Then $W = \int_C \vec{F} \cdot d\vec{r} = -\int_C \nabla P \cdot d\vec{r} \overset{FTPI}{=\!=} -(P(B) - P(A)) = P(A) - P(B)$

So, $W = P(A) - P(B)$

Ex. Newton's 2nd Law: $\vec{F} = m\vec{a} = \boxed{m\ddot{\vec{r}}(t)}$ $\qquad d\vec{r} = \dot{\vec{r}}\,dt$

Recall: $\vec{v}(t) = \dot{\vec{r}}(t)$

$v(t) = \|\vec{v}\| = \|\dot{\vec{r}}\|(t)$ $\qquad$ Also, $K(t) = \frac{1}{2}mv(t)^2$

$\vec{a}(t) = \ddot{\vec{r}}(t)$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C m\ddot{\vec{r}} \cdot \dot{\vec{r}}\,dt = m\int_C \ddot{\vec{r}} \cdot \dot{\vec{r}}\,dt$$

$$= m\int_a^b \langle \ddot{x}, \ddot{y}, \ddot{z} \rangle \cdot \langle \dot{x}, \dot{y}, \dot{z} \rangle\,dt$$

$$= m\int_a^b \ddot{x}\dot{x}\,dt + m\int_a^b \ddot{y}\dot{y}\,dt + m\int_a^b \ddot{z}\dot{z}\,dt$$

$u = \dot{x}$

$du = \ddot{x}\,dt$

$$= m\int_{\square}^{\square} u\,du = \frac{1}{2}mu^2 \Big|_{\square}^{\square}$$

Then, $W = K(B) - K(A)$

$$= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\Big|_a^b$$

$$= \frac{1}{2}m\|\dot{\vec{r}}\|^2\Big|_a^b = \frac{1}{2}mv^2(t)\Big|_a^b = K(B) - K(A)$$

All together

$$W = P(A) - P(B) = K(B) - K(A)$$

$$K(A) + P(A) = K(B) + P(B)$$

This proves the Conservation of Energy Law! $\blacksquare$