M344 6 Nov '18

Path Integrals:

$$\int_a^b f(x,y)\,ds$$
$$\int_a^b \vec{F}\cdot d\vec{r}$$

Fundamental Theorem for Path Integrals (FTPI)

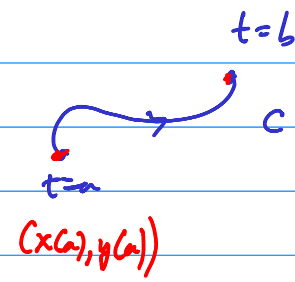
$\vec{F}$ is conservative $\Rightarrow$ $\vec{F} = \nabla f$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Proof in 2D. $C: \vec{r} = \langle x(t), y(t)\rangle \quad a \le t \le b$

$\vec{F} = \nabla f = \langle \partial_x f, \partial_y f\rangle$

$d\vec{r} = \langle dx, dy\rangle = \langle \dot{x}\,dt, \dot{y}\,dt\rangle$



$$\int_C \vec{F}\cdot d\vec{r} = \int_C \langle \partial_x f, \partial_y f\rangle \cdot \langle \underbrace{\dot{x}\,dt}_{dx}, \underbrace{\dot{y}\,dt}_{dy}\rangle$$

$$= \int_{x(a)}^{x(b)} \frac{\partial}{\partial x} f(x,y)\,dx + \int_{t=a}^{b} \frac{\partial}{\partial y} f(x,y)\,dy$$

$$\overset{FTC}{=} f(x,y)\Big|_{x(a)}^{x(b)} + f(x,y)\Big|_{y(a)}^{y(b)}$$

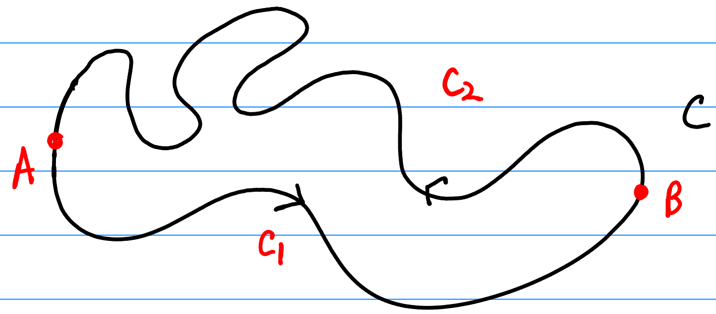
$$= f(x,y)\Big|_{(x(a),y(a))}^{(x(b),y(b))}$$

$$= f\big((x(b),y(b))\big) - f\big((x(a),y(a))\big)$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)). \qquad \blacksquare$$

Thm. $\int_C \vec{F}\cdot d\vec{r}$ is independent of path if and only if $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed path C.

Proof. Suppose the $\int_C \vec{F}\cdot d\vec{r}$ does not depend on the path C.



TBS: $\int_C \vec{F}\cdot d\vec{r} = 0$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1+C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = [f(B)-f(A)] + [f(A)-f(B)]$$

$$= f(B)\cancel{-f(B)} + f(A)\cancel{-f(A)}$$

$$= 0.$$

Thm. Suppose D is an open, connected region in $\mathbb{R}^n$.
Let $\vec{F}$ be a continuous vector field on D. $\vec{F} = \langle P(x,y), Q(x,y)\rangle$
If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path, then $\vec{F}$ is <u>conservative</u>: $\vec{F} = \nabla f$.

Conservative = Independence of Path

Q. How do we know if our v.f. is conservative?
In $\mathbb{R}^2$;

Thm. If $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is conservative, and P and Q have continuous first partial derivatives, then $\vec{F}$ is conservative if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Recall. If $\vec{F}$ is conservative, then $\vec{F} = \langle P, Q \rangle = \langle \partial_x f, \partial_y f \rangle$

So,

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

by Clairaut's Theorem $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ since they are continuous.

Ex. $\vec{F}(x,y) = (x-y)\vec{i} + (x-2)\vec{j}$

Is it conservative? $P(x,y) = x-y$, $Q(x,y) = x-2$

$$\left.\begin{aligned} \frac{\partial P}{\partial y} &= -1 \\ \frac{\partial Q}{\partial x} &= 1 \end{aligned}\right\} \neq$$

NOT conservative, so not path-independent, FTPI does not apply.

Ex. $\vec{F}(x,y) = \underbrace{(3+2xy)}_{\partial_x f}\vec{i} + \underbrace{(x^2-3y^2)}_{\partial_y f}\vec{j}$

Q. Is it conservative? What is potential func?

$P(x,y) = 3+2xy$

$Q(x,y) = x^2-3y^2$

$$\left.\begin{aligned} \frac{\partial P}{\partial y} &= 2x \\ \frac{\partial Q}{\partial x} &= 2x \end{aligned}\right\} =!$$

Conservative!

$$f = \int \frac{\partial f}{\partial x}\, dx = \int P(x,y)\, dx = \int 3+2xy\, dx = 3x + x^2 y + C_1(y)$$

$$f = \int \frac{\partial f}{\partial y}\, dy = \int x^2 - 3y^2\, dy = x^2 y - y^3 + C_2(x)$$

$$\boxed{f(x,y) = 3x + x^2 y - y^3}$$ Pot. Fun.

Ex. $\vec{F} = \langle 3 + 2xy,\ x^2 - 3y^2 \rangle$ $\quad$ $\vec{r}(t) = \langle e^t \sin t,\ e^t \cos t \rangle$, $0 \le t \le \pi$

$$\int_C \vec{F} \cdot d\vec{r} = ? \overset{FTP}{=} f(\vec{r}(\pi)) - f(\vec{r}(0))$$

$$\vec{r}(\pi) = \langle e^{\pi} \sin \pi,\ e^{\pi} \cos \pi \rangle = \langle 0, -e^{\pi} \rangle$$

$$\vec{r}(0) = \langle e^{0} \sin 0,\ e^{0} \cos 0 \rangle = \langle 0, 1 \rangle$$

$$\boxed{f(x,y) = 3x + x^2 y - y^3}$$

$$= f(0, -e^{\pi}) - f(0,1)$$

$$= \left[3(0) + 0^2 {-e^{\pi}} - (-e^{\pi})^3 \right] - \left[3 \cdot 0 + 0^2 \cdot 1 - 1^3 \right]$$

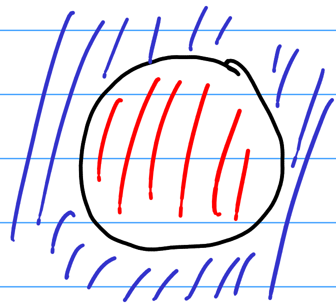
$$= \boxed{e^{3\pi} + 1}$$

§16.4 Green's Theorem

Thm (Jordan Curve Theorem)

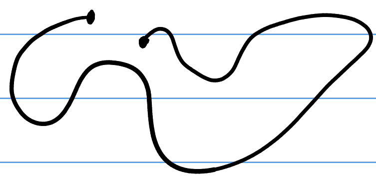
Every simply closed curve in the plane divides the plane into two parts: one bounded and one unbounded.

Ex.

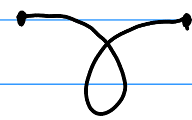


Simple and closed

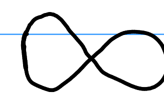
Simple, not closed



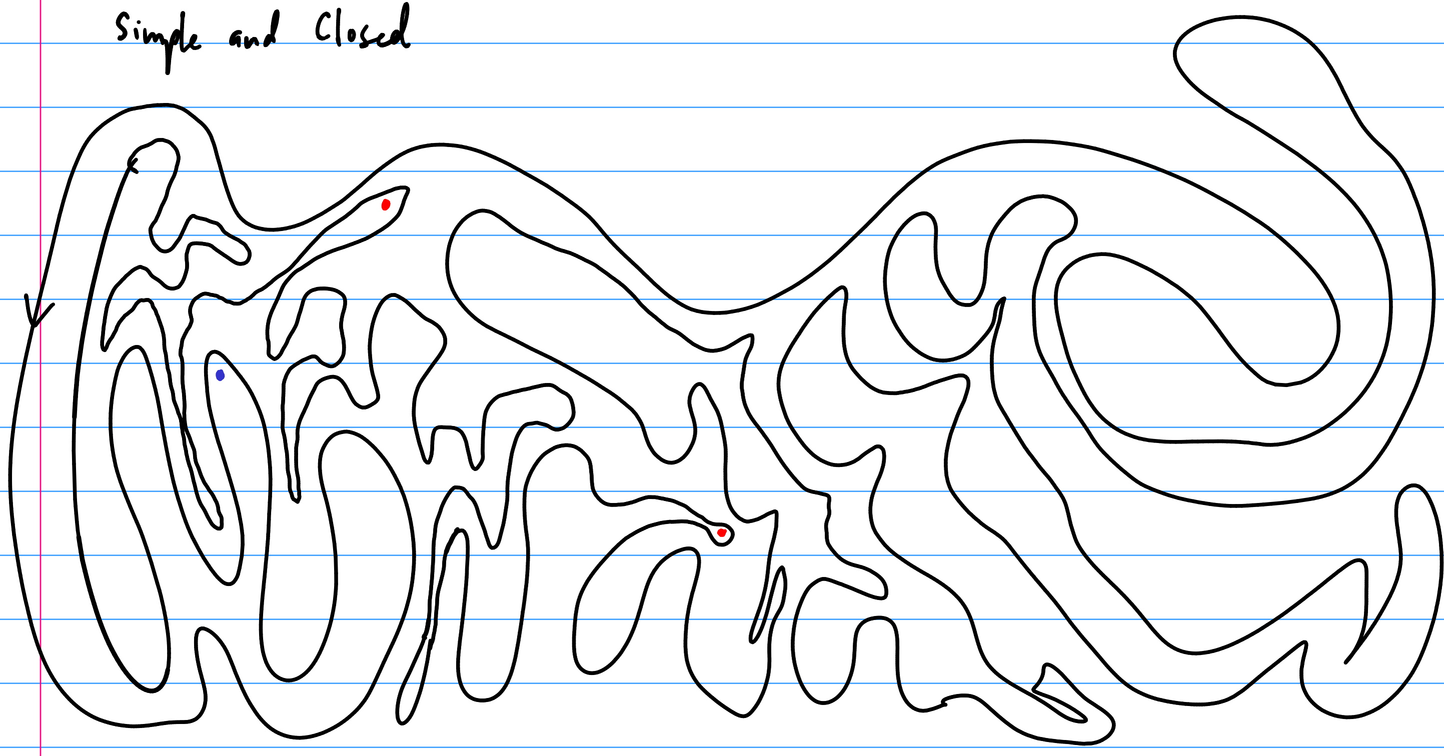
Not simple, Not closed



Closed, not simple



Simple and Closed



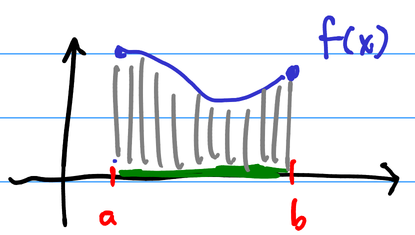
Thm. (Green's Theorem)
Let C be a positively-oriented Jordan curve, piecewise smooth, and let D denoted the bounded domain enclosed by C.
If $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ has continuous first partial derivatives in D, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Sometimes we denote $C = \partial D$ "boundary of D"

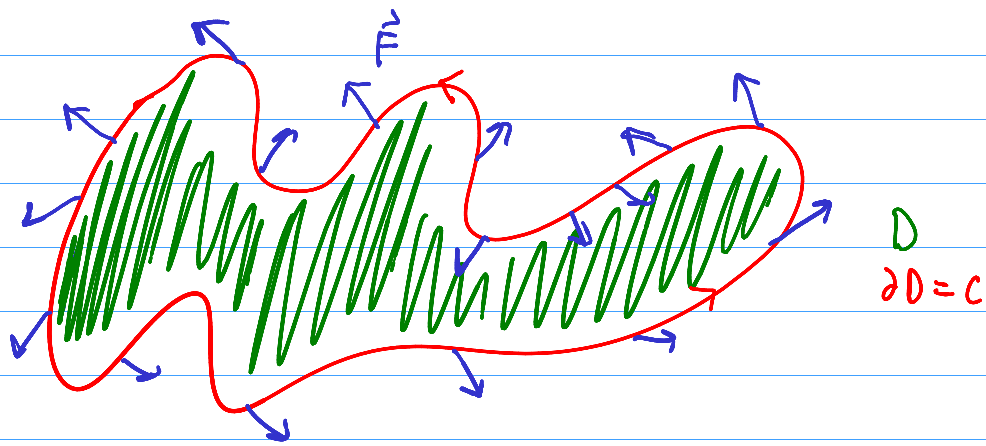
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P\,dx + Q\,dy$$

This is a 2D version of the FTC!



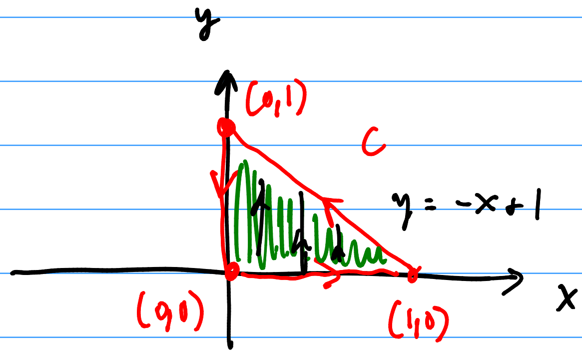
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$D = [a,b]$ $\quad$ $\partial D = C = \{a\} \cup \{b\}$



$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

<u>Ex</u>. $\int_C x^4\,dx + xy\,dy$ $\qquad$ C = triangle w/ vertices $(0,0), (0,1), (1,0)$



$P(x,y) = x^4 \qquad \frac{\partial P}{\partial y} = 0$

$Q(x,y) = xy \qquad \frac{\partial Q}{\partial x} = y$

$$\int_C x^4\,dx + xy\,dy = \iint_D y - 0\,dA$$

$$= \int_0^1 \int_0^{1-x} y\,dy\,dx$$

$$= \int_0^1 \frac{1}{2}(1-x)^2\,dx$$

$$= -\frac{1}{6}(1-x)^3\Big|_0^1 = \frac{1}{6}$$