M344 8 Nov '18

Ex. $\vec{F} = (3y - e^{\sin x})\vec{i} + (7x + \sqrt{y^4+1})\vec{j}$

$C: x^2+y^2 = 9$ (r^2)
$D: x^2+y^2 \le 9$

$\int_C \vec{F}\cdot d\vec{r} = ?$

$\vec{F}$ is continuous, check $\frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}$

$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(7x + \sqrt{y^4+1}\right) = 7$

$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(3y - e^{\sin x}\right) = 3$

Green's Thm: $\underset{\uparrow \text{gross}}{\int_C \vec{F}\cdot d\vec{r}} = \underbrace{\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA}$

$= \iint_D 7-3\, dA$

$= 4\iint_D dA = 4(\pi r^2) = \boxed{36\pi}$

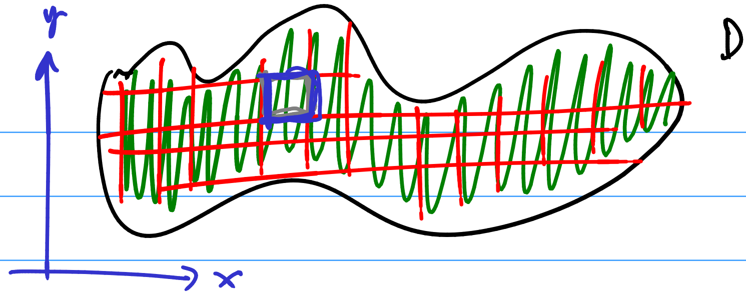
Ex. ~~Thought Experiment~~

$\text{Area}(D) = \iint_D 1\, dA \overset{?}{=} \underbrace{\int_{\partial D} \vec{F}\cdot d\vec{r}}$ Find $\vec{F}$ that works.

Green: $\iint_D \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{1} dA = \int_{\partial D} \vec{F}\cdot d\vec{r}$, $\vec{F} = \langle P, Q\rangle$

$\left.\begin{aligned} \frac{\partial Q}{\partial x} &= 1 \\ \frac{\partial P}{\partial y} &= 0 \end{aligned}\right\}$ $Q = x$, $P = 0$ $\vec{F} = \langle 0, x\rangle$

$\text{Area}(D) = \int_{\partial D} \langle 0,x\rangle\cdot\langle dx, dy\rangle = \boxed{\int_{\partial D} x\, dy}$



$A = l \cdot w$

$dA = x \cdot dy$

Another version?

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ $\qquad \frac{\partial Q}{\partial x} = 0 \qquad Q = 0$

$1 - 0$

$0 - (-1)$ $\qquad \frac{\partial P}{\partial y} = -1 \qquad P = -y$

$\vec{F} = \langle -y, 0 \rangle$

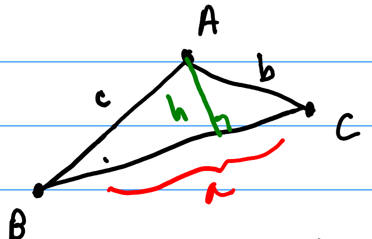
$$\boxed{A = -\int_{\partial D} y\,dx = \int_{\partial D} x\,dy}$$ or any linear combo.

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ $\qquad Q = 7x, \quad P = 6y \qquad \vec{F} = \langle 6y, 7x \rangle \qquad d\vec{r} = \langle dx, dy \rangle$

$7 - 6$

$$A = \int_{\partial D} 6y\,dx + 7x\,dy = 6\int_{\partial D} y\,dx + 7\int_{\partial D} x\,dy$$

Ex.



$A = \frac{1}{2} a h$

$A = \frac{1}{2} ac \sin B$

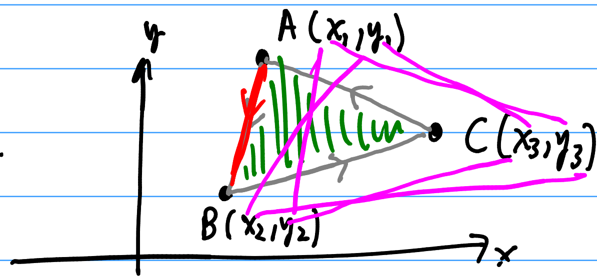
Heron's Formula:

$s = \frac{1}{2}(a+b+c)$

$A = \sqrt{s(s-a)(s-b)(s-c)}$

$\sin B = \frac{h}{c} \Rightarrow h = c \sin \theta$

Shoelace Formula:



$\underbrace{-\cancel{x_1 y_1} + \cancel{x_2 y_2}} \quad \underbrace{-\cancel{x_2 y_2} + \cancel{x_3 y_3}} \quad \underbrace{-\cancel{x_3 y_3} + \cancel{x_1 y_1}}$

$$A = \underbrace{\int_{AB} x\,dy} + \int_{BC} x\,dy + \int_{CA} x\,dy = \frac{1}{2}\left[x_1 y_2 - y_1 x_2 + x_2 y_3 - y_2 x_3 + x_3 y_1 - y_3 x_1\right]$$

$$\vec{r}_{AB} = (1-t)\vec{A} + t\vec{B} = (1-t)\langle x_1, y_1\rangle + t\langle x_2, y_2\rangle \qquad t: 0\to 1$$

$$= \langle x_1 + (x_2-x_1)t,\ y_1 + (y_2-y_1)t\rangle$$

$$x(t) = x_1 + (x_2-x_1)t \qquad dy = (y_2-y_1)\,dt$$

$$\int_{AB} x\,dy = \int_0^1 (x_1+(x_2-x_1)t)(y_2-y_1)\,dt$$

$$= \int_0^1 x_1(y_2-y_1) + (x_2-x_1)(y_2-y_1)t \; dt$$

$$= x_1(y_2-y_1)t + \frac{1}{2}(x_2-x_1)(y_2-y_1)t^2 \Big|_0^1$$

$$= x_1(y_2-y_1) + \frac{1}{2}x_2(y_2-y_1) - \frac{1}{2}x_1(y_2-y_1)$$

$$= \frac{1}{2}(x_1+x_2)(y_2-y_1) = \underbrace{\frac{1}{2}(x_1y_2 - x_1y_1 + x_2y_2 - x_2y_1)}$$

$$\cancel{= \frac{1}{2}(x_1y_2 - y_1x_2)}$$

Ex. Find the area of the ellipse $\left(\underset{u}{\frac{x}{a}}\right)^2 + \left(\underset{v}{\frac{y}{b}}\right)^2 = 1$.

$u^2 + v^2 = 1$

$\frac{x}{a} = u = \cos t$

$\frac{y}{b} = v = \sin t$

or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

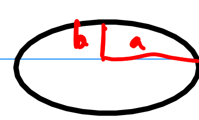
$$A = \int_C x\,dy$$

parametrize: $x(t) = a\cos t$, $y(t) = b\sin t$, $0 \le t \le 2\pi$

$$A = \int_0^{2\pi} a\cos t \cdot b\cos t\,dt \qquad dy = b\cos t\,dt$$

$$= ab\int_0^{2\pi} \cos^2 t\,dt = \frac{ab}{2}\int_0^{2\pi} 1 + \cos(2t)\,dt = \frac{ab}{2}\left(t + \cancel{\frac{1}{2}\sin(2t)}\right)\Big|_0^{2\pi}$$

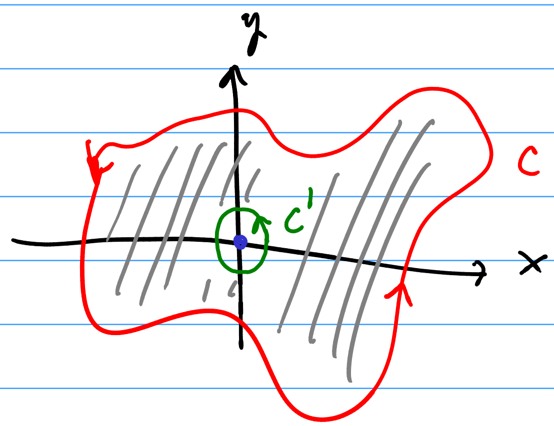
$$A = \frac{2ab\pi}{2} = ab\pi$$



$A = \pi ab$

Ex. $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

Compute $\int_C \vec{F}\cdot d\vec{r}$ around any closed curve C that encloses the origin



$$\frac{\partial P}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$ NOT cont. at $(0,0)$.

Take C' to be a circle w/ small radius so that it is contained inside C: enclosed by C.

Then D to be the bounded region between C and C'.

$$\partial D = C \cup C'$$

$$\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

$$\int_{\partial D} \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} - \int_{C'} \vec{F}\cdot d\vec{r}$$

Green's Thm does apply inside this D since we cut out the origin

$$\frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \frac{\partial Q}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\int_{\partial D} \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint_D 0\, dA = 0$$

Therefore,

$$\int_C \vec{F}\cdot d\vec{r} - \int_{C'} \vec{F}\cdot d\vec{r} = 0 \Rightarrow \overbrace{\int_C \vec{F}\cdot d\vec{r}}^{\text{any closed curve}} = \overbrace{\int_{C'} \vec{F}\cdot d\vec{r}}^{\text{unit circle}}$$

TBC: $\int_{C'} \vec{F}\cdot d\vec{r}$ $\quad \vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ $\quad C': \vec{r} = \langle \cos t, \sin t \rangle$

$0 \le t \le 2\pi$

$$x^2+y^2 = \cos^2 t + \sin^2 t = 1$$

$$\vec{F} = \langle -\sin t, \cos t \rangle \qquad d\vec{r} = \langle -\sin t, \cos t \rangle \, dt$$

So $\int_C \vec{F}\cdot d\vec{r} = \int_{C'} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = \int_0^{2\pi} 1 \, dt = 2\pi$

So, $\int_C \vec{F}\cdot d\vec{r} = 2\pi$ for all C enclosing the origin.
