M344 13 Nov 18

§16.5 : Div, Grad, Curl and all that (book, Google)

Let $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^3$.
$f = f(x,y,z)$

The gradient of f is the vector field

$$\text{grad}(f) = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \underbrace{\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle}\, f$$

$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ is the gradient operator : ∇: functions $\to$ vector fields

Let $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

or $\vec{F} = \langle P, Q, R \rangle$.

Defn. The curl of $\vec{F}$ is the vector field defined by

$$\text{curl}\,\vec{F} = \text{curl}(\vec{F}) = \nabla \times \vec{F}$$

$$\nabla \times \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

Ex. $\vec{F} = \langle 1, x+y, xy-\sqrt{z} \rangle$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 1 & x+y & xy-\sqrt{z} \end{vmatrix} = \langle x, -y, 1 \rangle$$

Thm. Let f be function in $\mathbb{R}^3$ such that all second partials are continuous.

Then $\text{curl}\,(\nabla f) = \vec{0}$

In other words, if $\vec{F} = \nabla f$ is conservative, then
$\text{curl}\,\vec{F} = \vec{0}$.

Proof. f, $\nabla f = \langle \partial_x f, \partial_y f, \partial_z f \rangle$
Now take the curl:

$$\text{Curl}(\nabla f) = \nabla \times (\nabla f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_x f & \partial_y f & \partial_z f \end{vmatrix}$$

$$= \langle \partial_{yz} f - \partial_{zy} f, \partial_{zx} f - \partial_{xz} f, \partial_{xy} f - \partial_{yx} f \rangle$$

Now by Clairaut's Thm, we get

$$= \langle 0, 0, 0 \rangle = \vec{0}. \qquad \blacksquare$$

Thm. If $\text{curl}\,\vec{F} = \vec{0}$, then $\vec{F}$ is conservative!

Together, $\text{curl}\,\vec{F} = \vec{0} \iff \vec{F}$ is conservative $\iff \vec{F} = \nabla f$

Ex. $\vec{F} = \langle 2xy,\ x^2 + 2yz,\ y^2 \rangle$ Conservative? & Potential function?

$\nabla f = \langle \partial_x f,\ \partial_y f,\ \partial_z f \rangle$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ 2xy & (x^2+2yz) & y^2 \end{vmatrix} = \langle 2y - 2y,\ 0-0,\ 2x - 2x \rangle$$

$$= \langle 0, 0, 0 \rangle = \vec{0}$$

Conservative!

$$\int \frac{\partial f}{\partial x}\, dx = \int 2xy\, dx \qquad f(x,y,z) = x^2y + \underline{C_1(y,z)}$$

$$f = \int x^2 + 2yz\ dy = x^2y + y^2z + \cancel{C_2(x,z)}$$

$$f = \int y^2\, dz = y^2z + \underline{C_3(x,y)}$$

So the potential function for this $\vec{F}$ is

$$\boxed{f(x,y,z) = x^2y + y^2z}$$

Ex. Let $\vec{F} = \langle P, Q \rangle$ (2D). Is $\vec{F}$ conservative? $\left\{ \begin{array}{l} P = P(x,y) \\ Q = Q(x,y) \end{array} \right.$

Extend $\vec{F}$ to $\vec{F} = \langle P, Q, 0 \rangle$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ P & Q & 0 \end{vmatrix} = \left\langle -\frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z},\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

$$= \left\langle 0, 0, \underline{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}} \right\rangle$$

So $\vec{F}$ is conservative if and only if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$,
or

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

This is the same criteria we found last week!

Defn. Let $\vec{F} = \langle P, Q, R \rangle$ be a smooth vector field on $\mathbb{R}^3$. The divergence of $\vec{F}$ is the function defined by

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \langle \partial x, \partial y, \partial z \rangle \cdot \langle P, Q, R \rangle$$

$$= \partial_x P + \partial_y Q + \partial_z R \quad \leftarrow \text{a function!}$$

Ex. $\vec{F} = \langle 1, x+y, xy - \sqrt{z} \rangle$ find $\operatorname{div}(\vec{F})$.

$$\operatorname{div} \vec{F} = \partial x(1) + \partial_y(x+y) + \partial_z(xy - \sqrt{z})$$

$$= 0 + 1 - \frac{1}{2\sqrt{z}}$$

$$\operatorname{div} \vec{F} = 1 - \frac{1}{2\sqrt{z}}.$$

Thm. If $\vec{F} = \langle P, Q, R \rangle$ is vector field such that P, Q, R have continuous 2^{nd} derivatives, then

$$\operatorname{div}(\operatorname{curl} \vec{F}) = 0$$

Proof. FTIS.

Start w/ $\operatorname{curl} \vec{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$, then take div.

Def'n. Let f be a smooth function in $\mathbb{R}^3$ w/ continuous 2^{nd} derivatives. The Laplacian of f is the function defined by

$$\Delta f = \operatorname{div}(\operatorname{grad} f) = \nabla \cdot \nabla f = \nabla^2 f$$

$f = f(x,y,z)$

$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

$\nabla \cdot \nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

$$\boxed{\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}}$$

Def'n. A function f is said to be harmonic if $\Delta f = 0$.

Ex. $f(x,y) = e^x \sin y$ is harmonic.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \stackrel{?}{=} 0$$

$\frac{\partial f}{\partial x} = e^x \sin y \qquad \frac{\partial^2 f}{\partial x^2} = e^x \sin y$

$\frac{\partial f}{\partial y} = e^x \cos y \quad + \quad \frac{\partial^2 f}{\partial y^2} = -e^x \sin y$

$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ ✓

<u>Ex</u> (FTIS) $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ in $\mathbb{R}^2$ in rectangular coords.

Write Δf in polar coords. ☺

Green's Theorem, Revisited

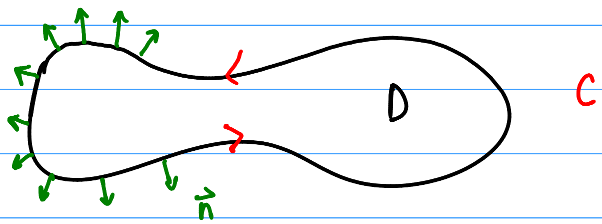
$\vec{F} = \langle P, Q \rangle$ a vector field
Domain D bounded by C, w/ $C: \vec{r}(t)$.

<u>Thm.</u> $\int_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

This thm can be rewritten in various forms:

1. $\int_C \vec{F} \cdot d\vec{r} = \iint_D (\text{curl}\, \vec{F}) \cdot \vec{k}\, dA$ w/ $\vec{k} = \langle 0, 0, 1 \rangle$.
   and $\vec{F} = \langle P, Q, 0 \rangle$

2. The normal vector field along the boundary C is the unit vector field orthogonal to $\vec{T}$, pointing outward from the bounded region.



If $\vec{r} = \langle x(t), y(t) \rangle$, then $\vec{n}(t) = \frac{1}{\|\dot{\vec{r}}(t)\|} \langle \dot{y}(t), -\dot{x}(t) \rangle$

<u>Green's Thm:</u> $\int_C \vec{F} \cdot \vec{n}\, ds = \iint_D \text{div}\, \vec{F}\, dA$

Proof: $\vec{F} = \langle P, Q \rangle$, $\underbrace{\vec{n} = \frac{1}{\|\dot{\vec{r}}\|} \langle \dot{y}, -\dot{x} \rangle}$, $ds = \|\dot{\vec{r}}\|\, dt$

$$\int_C \langle P, Q \rangle \cdot \langle \dot{y}, -\dot{x} \rangle \frac{1}{\cancel{\|\dot{\vec{r}}\|}} \cancel{\|\dot{\vec{r}}\|}\, dt$$

$$= \int_C P \underbrace{\dot{y}\, dt}_{dy} - Q \underbrace{\dot{x}\, dt}_{dx} = \int_C (-Q)\, dx + P\, dy$$

$$= \int_C \langle -Q, P \rangle \cdot d\vec{r}$$ and usual Green's Theorem applies!

$$\overset{GT}{=} \iint_D \left(\frac{\partial P}{\partial x} - \frac{-\partial Q}{\partial y} \right) dA = \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\, dA$$

$$\boxed{\int_C \vec{F} \cdot \vec{n}\, ds = \iint_D \operatorname{div} \vec{F}\, dA}$$

On Final.

Green's Identities : Integration by Parts.

NEXT.