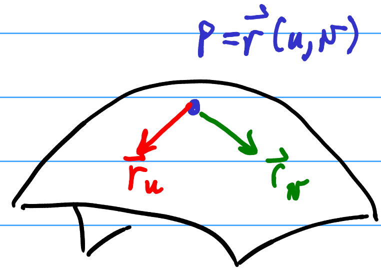
M344 — 20 Nov '18

§16.6, cont'd — Parametrized Surfaces

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

Graph: $S = \Gamma(\vec{r})$



$\vec{r}_u = \dfrac{\partial \vec{r}}{\partial u} = \langle \partial_u x, \partial_u y, \partial_u z \rangle$ at P

$\vec{r}_v = \dfrac{\partial \vec{r}}{\partial v} = \langle \partial_v x, \partial_v y, \partial_v z \rangle$

Def'n. A surface is said to be smooth iff $\vec{r}_u \times \vec{r}_v \neq \vec{0}$.

Ex. $\vec{r} = \langle u^2, v^2, u+2v \rangle$

Find an eq'n of the tan. plane at $P(1,1,3)$

$$\left.\begin{matrix} u^2 = 1 \\ v^2 = 1 \\ u+2v = 3 \end{matrix}\right\} \quad \begin{matrix} u = \pm 1 \\ v = \pm 1 \\ (u,v) = (1,1) \end{matrix}$$

The normal vector to tangent plane is $\vec{n} = \vec{r}_u \times \vec{r}_v$.

$(u,v) = (1,1)$

$\vec{r}_u = \langle 2u, 0, 1 \rangle = \langle 2, 0, 1 \rangle$

$\vec{r}_v = \langle 0, 2v, 2 \rangle = \langle 0, 2, 2 \rangle$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ 0 & 2 & 2 \end{vmatrix} = \langle -2, -4, 4 \rangle$$

$\Pi: \ ax + by + cz + d = 0$

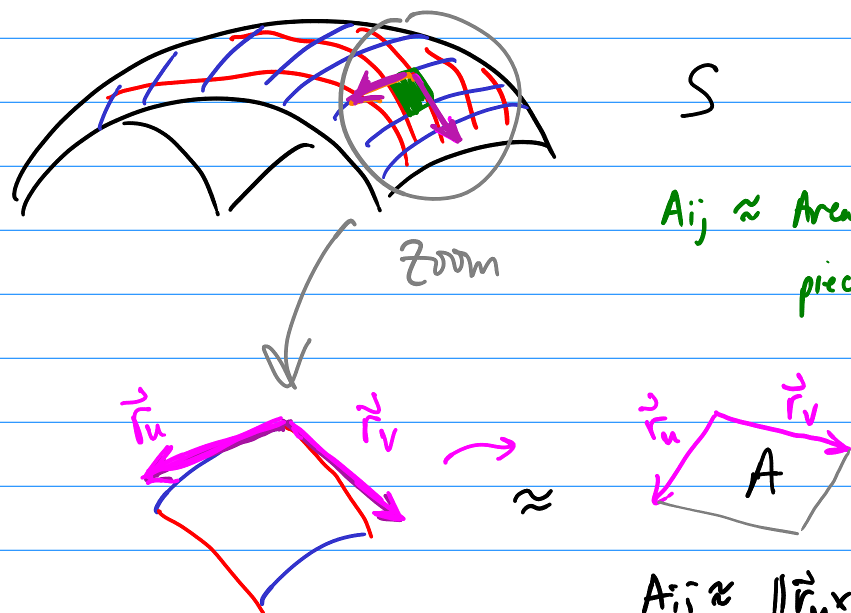
$d = -\vec{n} \cdot \vec{P}$

$\langle 2, 4, -4 \rangle \cdot \langle 1, 1, 3 \rangle$

$d = 2 + 4 - 12 = -6$

All together now, π : $\boxed{-2x - 4y + 4z - 6 = 0}$

Given a surface, can we compute its surface area?



$A_{ij} \approx$ Area of a small piece of tan. plane

$A_{ij} \approx \|\vec{r}_u \times \vec{r}_v\|$

<u>Defn.</u> Let S be a surface parametrized by $\vec{r}(u,v)$ where $\vec{r}$ to covers S only once, and (u,v) lies in the domain D.

Then the surface area is

$$\boxed{A(S) = \iint_D \|\vec{r}_u \times \vec{r}_v\| \, dA}$$

<u>Ex.</u> Find the surface area of the sphere $x^2 + y^2 + z^2 = r^2$.

<u>parametrization</u>;
$$\begin{cases} x = r \sin\varphi \cos\theta & 0 \le \varphi \le \pi \\ y = r \sin\varphi \sin\theta & 0 \le \theta < 2\pi \\ z = r \cos\varphi & \end{cases}$$

Compute $\vec{r}_\varphi$, $\vec{r}_\theta$.

$\vec{r}_\varphi = \langle r\cos\varphi\cos\theta, r\cos\varphi\sin\theta, -r\sin\varphi \rangle$

$$\vec{r}_\theta = \langle -r\sin\varphi\sin\theta,\ r\sin\varphi\cos\theta,\ 0\rangle$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r\cos\varphi\cos\theta & r\cos\varphi\sin\theta & -r\sin\varphi \\ -r\sin\varphi\sin\theta & r\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= \langle +r^2\sin^2\varphi\cos\theta,\ r^2\sin^2\varphi\sin\theta,\ \underbrace{r^2\sin\varphi\cos\varphi\cos^2\theta + r^2\sin\varphi\cos\varphi\sin^2\theta}_{r^2\sin\varphi\cos\varphi}\rangle$$

$$\|\vec{r}_u \times \vec{r}_v\|^2 = \underbrace{r^4\sin^4\varphi\cos^2\theta + r^4\sin^4\varphi\sin^2\theta}\ + r^4\sin^2\varphi\cos^2\varphi$$

$$= r^4\sin^2\varphi\sin^2\varphi + r^4\sin^2\varphi\cos^2\varphi$$

$$= r^4\sin^2\varphi$$

$$\|\vec{r}_u \times \vec{r}_v\| = r^2\sin\varphi$$

$$A(S) = \iint_D r^2\sin\varphi\, dA = \int_0^{\pi}\int_0^{2\pi} r^2\sin\varphi\, d\theta\, d\varphi$$

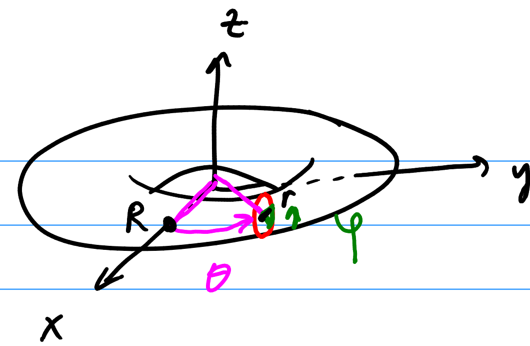
$$= r^2\int_0^{2\pi} d\theta \int_0^{\pi}\sin\varphi\, d\varphi$$

$$= r^2 \cdot 2\pi \cdot (-\cos\varphi)\Big|_0^{\pi}$$

$$= r^2(2\pi)(-(-1)+(1))$$

$$\boxed{A(S) = 4\pi r^2}\ !\ \ \text{:)}$$

Ex. Parametrized Torus:

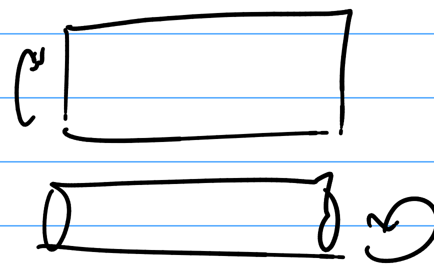


$0 < r << R$

$$\begin{cases} x = (R + r\cos\varphi)\cos\theta \\ y = (R + r\cos\varphi)\sin\theta \\ z = r\sin\varphi \end{cases}$$

$0 \leq \theta, \varphi \leq 2\pi$

Find the surface area.



$$\vec{r}_\varphi = \langle -r\sin\varphi\cos\theta, -r\sin\varphi\sin\theta, +r\cos\varphi \rangle$$

$$\vec{r}_\theta = \langle -(R + r\cos\varphi)\sin\theta, (R + r\cos\varphi)\cos\theta, 0 \rangle$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \langle -r(R + r\cos\varphi)\cos\varphi\cos\theta, +r(R + r\cos\varphi)\cos\varphi\sin\theta, -r(R + r\cos\varphi)\sin\varphi\cos^2\theta - r(R + r\cos\varphi)\sin\varphi\sin^2\theta \rangle$$

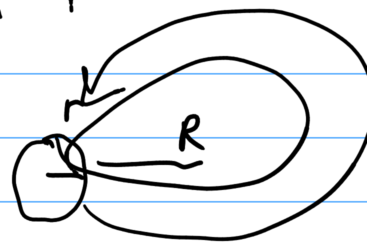
$$= \langle -r(R + r\cos\varphi)\cos\varphi\cos\theta, r(R + r\cos\varphi)\cos\varphi\sin\theta, -r(R + r\cos\varphi)\sin\varphi \rangle$$

$$\|\vec{r}_\varphi \times \vec{r}_\theta\| = r^2(R + r\cos\varphi)^2 \Big[\underbrace{\underbrace{\left[\cos^2\varphi\cos^2\theta + \cos^2\varphi\sin^2\theta\right]}_{\cos^2\varphi} + \sin^2\varphi}_{1}\Big] = r^2(R + r\cos\varphi)^2$$

TBI: $\|\vec{r}_\varphi \times \vec{r}_\theta\| = r(R + r\cos\varphi)$

$$A(\mathbb{T}^2) = \int_0^{2\pi}\int_0^{2\pi} rR + r^2\cos\varphi \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{2\pi} rR + \cancel{r^2\cos\varphi}^{\,0}\, d\varphi$$



$$= 2\pi r \cdot 2\pi R$$

$$= 4\pi^2 rR$$

Ex. $f(x,y)$ graph is a surface.

$$\vec{r}(x,y) = \langle x, y, f(x,y)\rangle$$

$$A(\Gamma(f)) = \iint_D \|\vec{r}_x \times \vec{r}_y\|\, dA$$

$$\underset{\text{FTIS}}{=} \iint_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\, dA$$

!