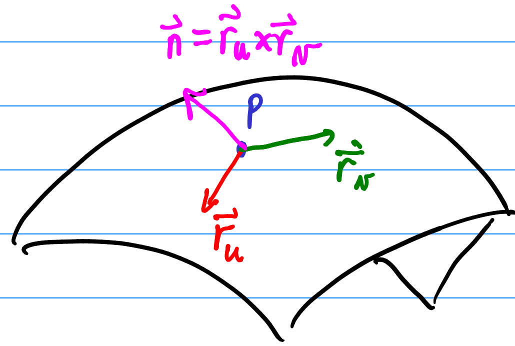
M344 27 Nov '18

Let S be a smooth surface.

$S: \vec{r}(u,v)$

$\vec{r}_u = \frac{\partial \vec{r}}{\partial u}$, $\vec{r}_v = \frac{\partial \vec{r}}{\partial v}$



$\vec{n}$ is continuous v.f. on $S \Leftrightarrow S$ is smooth.

Surface Area of $S = \text{Area}(S) = \iint_S 1\, dS = \iint_D \|\vec{n}\|\, dA$

$$= \iint_D \|\vec{r}_u \times \vec{r}_v\|\, dA$$

D is the domain of the parameters (u,v).

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§16.7 - Surface Integrals

Let f be a continuous function on $\mathbb{R}^3$ defined on a domain that contains the surface S. Let S be parametrized by (u,v) s.t. the surface is covered only once.

Then the integral of f along S is defined to be

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, \|\vec{r}_u \times \vec{r}_v\|\, dA$$

Think: 2D version of a path integral.

$$\int_C f(x,y)\, ds = \int_a^b f(\vec{r}(t))\, \|\dot{\vec{r}}(t)\|\, dt$$

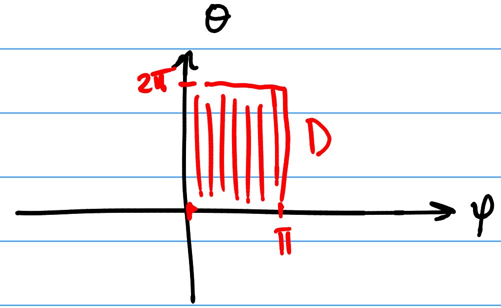
Ex. $\iint_S x^2 \, dS$ where S is the unit sphere.

$$= \iint_D x^2 \overbrace{\|\vec{r}_\theta \times \vec{r}_\varphi\|}^{\sin\varphi} \, dA \qquad \begin{cases} x = \sin\varphi\cos\theta \\ y = \sin\varphi\sin\theta \\ z = \cos\varphi \end{cases}$$

$dA = d\varphi \, d\theta$

$x^2 = \sin^2\varphi\cos^2\theta$

$0 \le \varphi \le \pi \,, \quad 0 \le \theta \le 2\pi$



$\vec{r}_\theta = \langle -\sin\varphi\sin\theta, \ \sin\varphi\cos\theta, \ 0 \rangle$

$\vec{r}_\varphi = \langle \cos\varphi\cos\theta, \ \cos\varphi\sin\theta, \ -\sin\varphi \rangle$

$$\vec{n} = \vec{r}_\theta \times \vec{r}_\varphi = \langle -\sin^2\varphi\cos\theta, \ -\sin^2\varphi\sin\theta, \ \underbrace{-\sin^2\theta\sin\varphi\cos\varphi - \cos^2\theta\sin\varphi\cos\varphi}_{-\sin\varphi\cos\varphi} \rangle$$

$$\|\vec{n}\|^2 = \underbrace{\sin^4\varphi\cos^2\theta + \sin^4\varphi\sin^2\theta}_{\sin^2\varphi\sin^2\varphi} + \sin^2\varphi\cos^2\varphi = \sin^2\varphi\,(\overset{1}{\cancel{\sin^2\varphi + \cos^2\varphi}})$$

so $\|\vec{n}\| = \sin\varphi$

and, $$\iint_S x^2 \, dS = \int_0^{2\pi}\int_0^{\pi} \sin^2\varphi\cos^2\theta \cdot \sin\varphi \, d\varphi \, d\theta$$

$$= \underbrace{\int_0^{2\pi} \cos^2\theta \, d\theta}_{} \cdot \int_0^{\pi} \underbrace{\sin^2\varphi}_{1-\cos^2\varphi} \sin\varphi \, d\varphi$$

$$= \frac{1}{2}\int_0^{2\pi} 1 + \cos(2\theta) \, d\theta \cdot \left[\int_0^{\pi} \sin\varphi \, d\varphi + \int_1^{-1} u^2 \, du \right]$$

$$= \frac{1}{\cancel{2}} \cdot \cancel{2}\pi \cdot \left[2 - \frac{2}{3} \right] = \boxed{\frac{4\pi}{3}}$$

Ex. Suppose the surface is the graph of a function $z=g(x,y)$.

$$\to \quad dS = \|\vec{n}\|\, dA = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,\underbrace{g(x,y)}_{z}) \sqrt{1+\underbrace{\left(\frac{\partial z}{\partial y}\right)^2+\left(\frac{\partial z}{\partial x}\right)^2}_{\|\nabla g\|^2}}\; dx\, dy$$

Ex. $\iint_S y\, dS$ $\qquad S=\Gamma(g)$, $\quad g(x,y)=x+y^2$, $\quad 0\le x\le 1$, $\quad 0\le y\le 2$.

$f(x,y,z)=y$

$$\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle$$

$$\nabla g = \langle 1, 2y \rangle$$

$$\|\nabla g\|^2 = 1+4y^2$$

$$dS = \sqrt{1+1+4y^2}\, dx\, dy = \sqrt{2+4y^2}\, dx\, dy$$

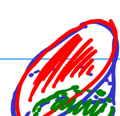
$$\iint_S y\, dS = \iint_D y\cdot\sqrt{2+4y^2}\, dx\, dy = \int_0^1 dx \cdot \frac{1}{8}\int_0^2 8y\sqrt{\underbrace{2+4y^2}_{u}}\, dy$$

$$= 1\cdot\frac{1}{8}\int_2^{18}\sqrt{u}\, du = \frac{1}{8}\cdot\frac{2}{3}u^{3/2}\Big|_2^{18}$$

$$= \frac{1}{12}\left(27\sqrt{2}^3-\sqrt{2}^3\right) = \frac{26}{12}\sqrt{2}^3$$

$$= \boxed{\frac{13\sqrt{2}^3}{6}}$$

Ex. $\iint_S z\, dS$ $\qquad S$ is the surface made up of 3 parts



$S_1: x^2+y^2=1$ $\qquad S_3: z=1+x$

$S_2: x^2+y^2\le 1$ in xy plane

$$\iint_S z\,dS = \iint_{S_1} z\,dS + \overbrace{\cancel{\iint_{S_2} z\,dS}}^{0} + \iint_{S_3} z\,dS$$

$S_1: \; x^2+y^2=1$ $\quad \begin{cases} x = \cancel{r}\cos\theta \\ y = \cancel{r}\sin\theta \\ z = z \end{cases}$, $r=1$ $\qquad (z,\theta)$

What is dS here?

$$\vec{r}_\theta = \langle -\sin\theta, \cos\theta, 0\rangle$$
$$\vec{r}_z = \langle 0, 0, 1\rangle$$

$$\vec{n} = \langle \cos\theta, \sin\theta, 0\rangle$$

$$\|\vec{n}\| = 1$$

$$\iint_{S_1} z\,dS = \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dz\,d\theta = \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \underbrace{\cos^2\theta}_{\underbrace{\frac{1}{2}(1+\cancel{\cos 2\theta})}_{\pi}} + \cancel{2\cos\theta} + \underbrace{1}_{2\pi}\,d\theta$$

$$\iint_{S_1} z\,dS = \frac{3\pi}{2}$$

$S_3: \; z = 1+x$ function! $\qquad dS = \sqrt{1+1}\,dx\,dy = \sqrt{2}\,dx\,dy$

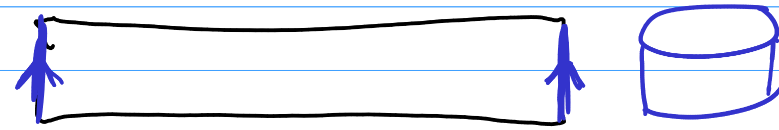
$$\iint_{D_3} (1+x)\sqrt{2}\,dA \overset{\text{Polar!}}{=} \int_0^{2\pi}\int_0^1 (1+r\cos\theta)\sqrt{2}\,r\,dr\,d\theta$$

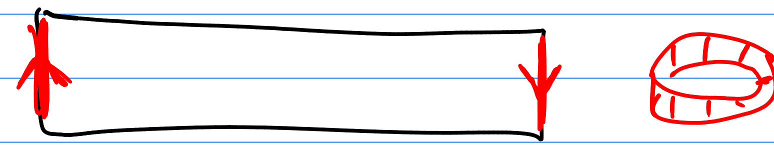
$$= \int_0^{2\pi}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{3}\cancel{\cos\theta}\right)d\theta = \sqrt{2}\,\pi$$

The total integral is: $\displaystyle\iint_S z\,dS = \frac{3\pi}{2} + 0 + \sqrt{2}\,\pi = \left(\frac{3+2\sqrt{2}}{2}\right)\pi$

Not all surfaces are created equally.

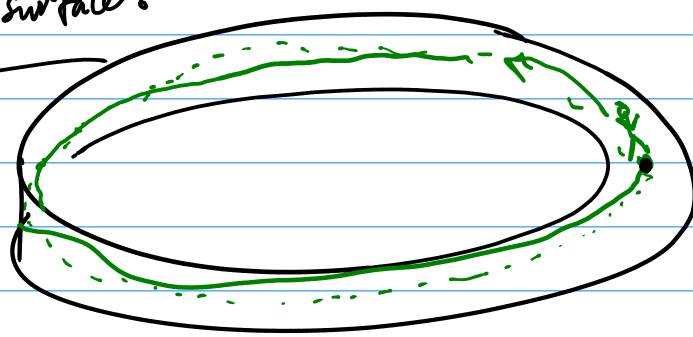
<u>Thought Experiment</u>:





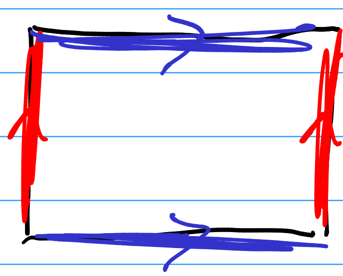
Möbius

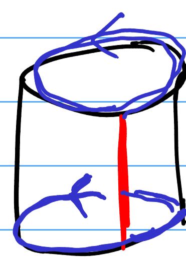
One-sided surface!

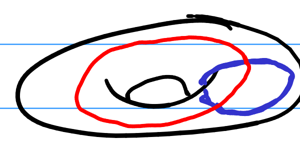


Non-orientable surface.

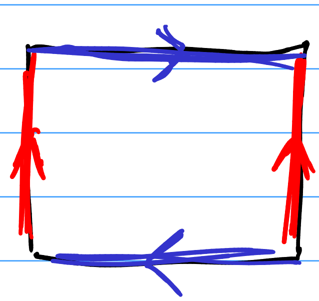
More experiments:

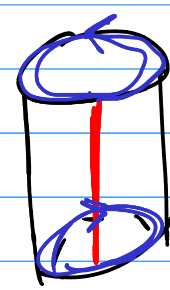




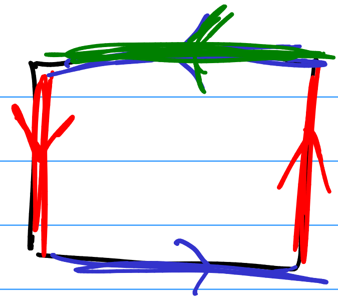


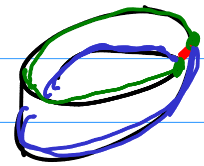
<u>Torus</u>

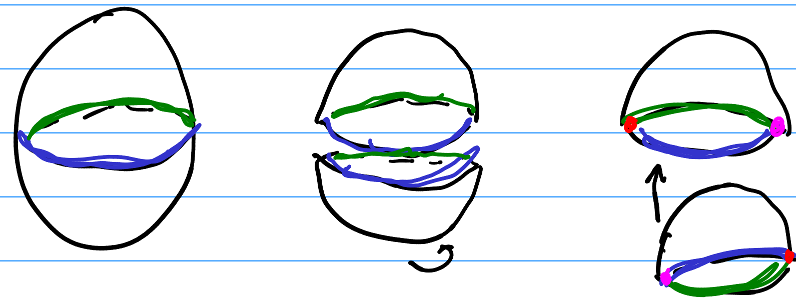


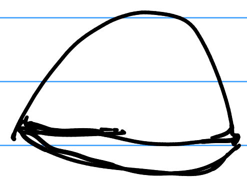


<u>Klein bottle</u>









$\mathbb{R}P^2$ projective plane