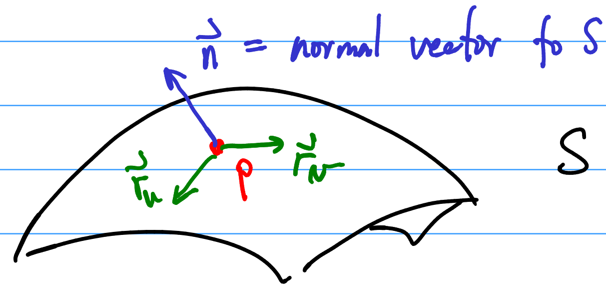
M344 29 Nov '18

§16.7 cont'd

Let S be a surface in $\mathbb{R}^3$



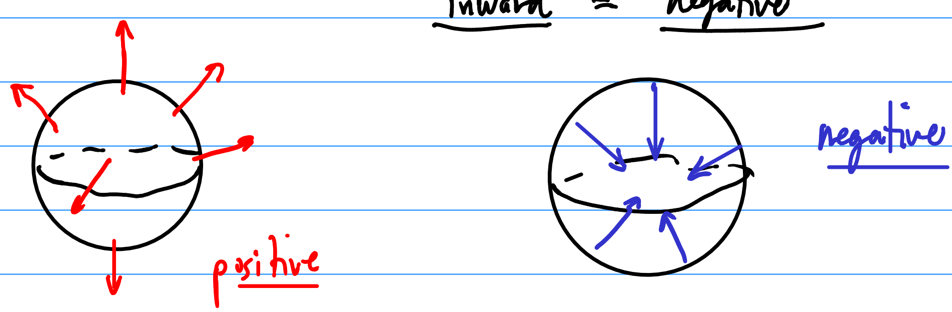
Orientable if we can choose a continuous vector field $\vec{n}$ on S.

The unit normal vector field on S will be

$$\boxed{\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}}$$

If S is closed, eg the sphere, then there is an unambiguous choice of orientation:

outward = positive
inward = negative



Suppose $\vec{F}$ is a vector field defined on some open region in $\mathbb{R}^3$ containing S. $\vec{F} = \langle P, Q, R \rangle$ are continuous on S.
S is orientable.

$d\vec{S} = \vec{n}\, dS$

Then,

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS}$$

$S: \ \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

$$dS = \|\vec{r}_u \times \vec{r}_v\| \, dA \qquad \text{where } dA \approx du\,dv$$

and

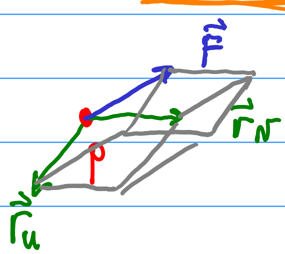
$$\vec{n} = \frac{(\vec{r}_u \times \vec{r}_v)}{\|\vec{r}_u \times \vec{r}_v\|}$$

Substituting in,

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, dS = \iint_D \vec{F} \cdot \frac{(\vec{r}_u \times \vec{r}_v)}{\cancel{\|\vec{r}_u \times \vec{r}_v\|}} \cancel{\|\vec{r}_u \times \vec{r}_v\|} \, dA$$

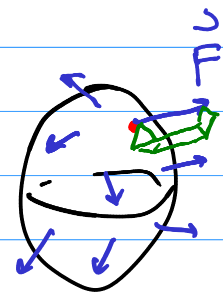
$$\boxed{= \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, dA}$$

Compute w/ this!



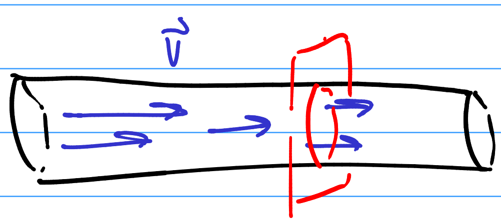
parallelepiped

$\vec{F} \cdot (\vec{r}_u \times \vec{r}_v) = $ volume!



$\iint_S \vec{F} \cdot d\vec{S}$ is the

<u>flux</u> <u>of</u> <u>$\vec{F}$</u> <u>across S</u>.



Ex. $\vec{F} = \langle z, y, x \rangle \qquad S: x^2+y^2+z^2=1 \quad$ unit sphere

Find the flux.

$$F = \iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\, dA \qquad \vec{F} = \vec{F}(u,v), \quad dA = du\,dv$$

$$\vec{r}\begin{cases} x = \sin\varphi\cos\theta \\ y = \sin\varphi\sin\theta \\ z = \cos\varphi \end{cases} \rightarrow \vec{F} = \langle \cos\varphi,\ \sin\varphi\sin\theta,\ \sin\varphi\cos\theta \rangle$$

$$\vec{r}_\varphi = \langle \cos\varphi\cos\theta,\ \cos\varphi\sin\theta,\ -\sin\varphi \rangle$$

$$\vec{r}_\theta = \langle -\sin\varphi\sin\theta,\ \sin\varphi\cos\theta,\ 0 \rangle$$

$$\rightarrow \vec{r}_\varphi \times \vec{r}_\theta = \langle \sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \cos\varphi\sin\varphi \rangle$$

$$\vec{F}\cdot(\vec{r}_\varphi \times \vec{r}_\theta) = \sin^2\varphi\cos\varphi\cos\theta + \sin^2\varphi\sin^2\theta\sin\varphi + \sin^2\varphi\cos\varphi\cos\theta$$

$$= \sin^2\varphi\left(\underline{\cos\varphi\cos\theta} + \sin^2\theta\sin\varphi + \underline{\cos\varphi\cos\theta}\right)$$

$$= \sin^2\varphi\left(\underline{2}\cos\varphi\,\underline{\cos\theta} + \sin^2\theta\sin\varphi\right)$$

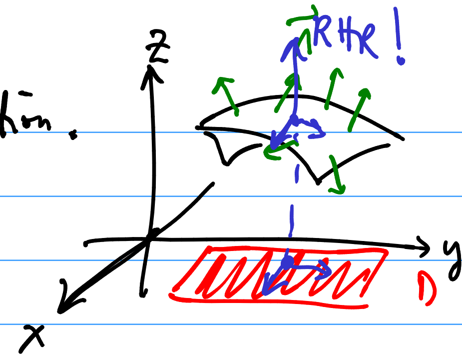
$$F = \int_0^{2\pi}\int_0^{\pi} 2\sin^2\varphi\cos\varphi\cos\theta\, d\varphi\, d\theta + \int_0^{2\pi}\int_0^{\pi} \sin^3\varphi\sin^2\theta\, d\varphi\, d\theta$$

$$= \cancel{\underbrace{\int_0^{2\pi}\cos\theta\, d\theta}_{0}\int_0^{\pi} 2\sin^2\varphi\cos\varphi\, d\varphi} + \underline{\int_0^{2\pi}\sin^2\theta\, d\theta}\ \underline{\int_0^{\pi}\sin^3\varphi\, d\varphi}$$

$$\underbrace{\frac{1}{2}\int_0^{2\pi} 1 - \cancel{\cos(2\theta)}^{\,0}\, d\theta}_{2\pi} \cdot \int_0^{\pi} (1-\cos^2\varphi)\,\underline{\sin\varphi\, d\varphi} \qquad \begin{aligned} u &= \cos\varphi \\ du &= -\sin\varphi\, d\varphi \end{aligned} \quad \begin{aligned} u(\pi) &= -1 \\ u(0) &= 1 \end{aligned}$$

$$= \pi \cdot \int_{-1}^{1} 1-u^2\, du = 2\pi\left(u - \tfrac{1}{3}u^3\right)\Big|_0^1 = 2\pi\left(1-\tfrac{1}{3}\right) = \boxed{\frac{4\pi}{3}}$$

Ex. $S = \Gamma(g)$ $\quad z = g(x,y)$ graph of a function.



$\vec{F} = \langle P, Q, R \rangle$

$\vec{r} = \langle x, y, g(x,y) \rangle$

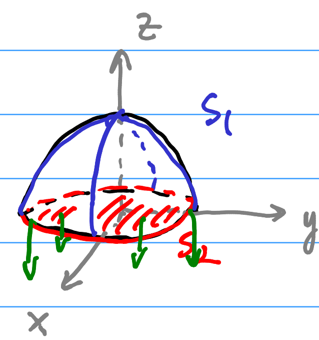
$$\left.\begin{aligned} \vec{r}_x &= \langle 1, 0, \tfrac{\partial g}{\partial x} \rangle \\ \vec{r}_y &= \langle 0, 1, \tfrac{\partial g}{\partial y} \rangle \end{aligned}\right\} \quad \vec{r}_x \times \vec{r}_y = \left\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \right\rangle$$

$$\vec{F}\cdot(\vec{r}_x \times \vec{r}_y) = \underbrace{-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R}$$

$$F = \iint_{\Gamma(g)} \vec{F}\cdot d\vec{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \longleftarrow$$

Ex. $\vec{F} = \langle y, x, \boxed{z} \rangle$

S: $\underbrace{z = 1 - x^2 - y^2}_{S_1} \cup \underbrace{z = 0}_{S_2}$



$x^2 + y^2 = 1$

S_2: $x^2 + y^2 \le 1$: $\vec{r} = \langle r\cos\theta, r\sin\theta, 0 \rangle$ $\quad 0 \le r \le 1$

$0 \le \theta < 2\pi$

$$\left.\begin{aligned} \vec{r}_r &= \langle \cos\theta, \sin\theta, 0 \rangle \\ \vec{r}_\theta &= \langle -r\sin\theta, r\cos\theta, 0 \rangle \end{aligned}\right\} \quad \vec{n} = \langle 0, 0, r \rangle \sim \underbrace{\langle 0, 0, -r \rangle}_{\vec{n}_2}$$

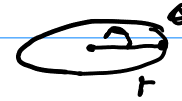
$$\vec{F}|_{S_2} = \langle y, x, z \rangle = \langle y, x, 0 \rangle$$

$$F_2 = \iint_D \vec{F}|_{S_2}\cdot\vec{n}_2 \, dA = \iint_D 0 + 0 + 0 \, dA = 0. \quad \checkmark$$

$\vec{F} = \langle y, x, \boxed{z} \rangle$

$\boxed{S: \ z = 1 - x^2 - y^2}$

$$F_1 = \iint_D +2yx + 2xy + 1 - x^2 - y^2 \, dA$$



$$= \int_0^{2\pi}\int_0^1 \left(4r^2 \sin\theta\cos\theta + 1 - r^2\right) r \, dr\, d\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$$= \int_0^{2\pi} \left(r^4 \sin\theta\cos\theta + \tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\right)\Big|_{r=0}^{r=1}$$

$$= \cancel{\int_0^{2\pi} \underbrace{\sin\theta}_{u} \underbrace{\cos\theta\, d\theta}_{du}} + \underbrace{\int_0^{2\pi} \tfrac{1}{4}\, d\theta}_{\pi/2}$$

$$0 + \pi/2$$

So, the total flux is $\boxed{F = \iint_S \vec{F}\cdot d\vec{S} = \pi/2}$

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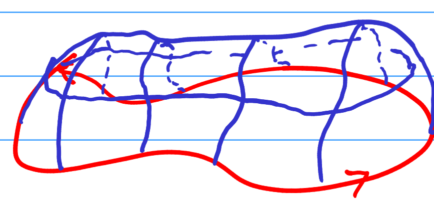
§16.8 – Stokes' Theorem

Thm – Let S be an oriented surface, piecewise smooth, bounded by a simple closed curve C w/ positive orientation.



S

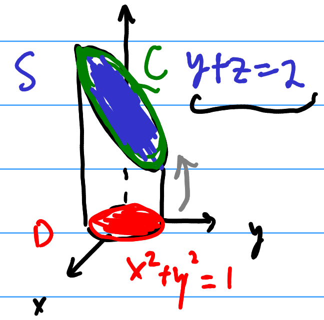
C

Let $\vec{F}$ be a vector field whose components have continuous first derivatives on S. Then

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\, dS}$$

Ex. Compute $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F}=\langle -y^2, x, z^2\rangle$ where C

is the intersection of $x^2+y^2=1$ and $y+z=2$.



$$\int_C \vec{F}\cdot d\vec{r} = \underbrace{\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dS}$$

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \times \vec{F}$$

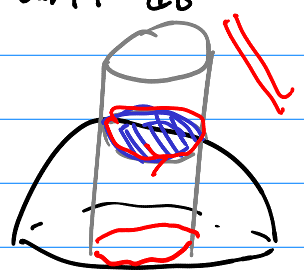
$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle$$
$$\vec{F} = \langle -y^2, x, z^2\rangle$$
$$\operatorname{curl}\vec{F} = \langle 0-0, 0-0, 1+2y\rangle = \langle 0,0,1+2y\rangle = \langle P, Q, R\rangle$$

$$g(x,y) = 2-y$$

So $\displaystyle\iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA = \iint_D 0-0+1+2y \; dA$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D 1+2y\,dA = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)r\,dr\,d\theta$$
$$= \int_0^{2\pi} \frac{1}{2} + \cancel{\frac{2}{3}\sin\theta}\,d\theta$$
$$= \boxed{\pi}$$

Ex. $\displaystyle\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$ $\quad \vec{F} = \langle xz, yz, xy\rangle$ $\quad S:$ $\underbrace{x^2+y^2}+z^2=4$ inside $\underbrace{x^2+y^2=1}$ $z>0$



$$= \int_C \vec{F}\cdot d\vec{r}$$

$$\vec{r} = \begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = k = \sqrt{3} \end{cases} \qquad d\vec{r} = \langle -\sin\theta, \cos\theta, 0 \rangle$$

<u>Finish</u>: <u>get 0</u>.