DGS
GPC3

Calculus III with Analytic Geometry
Good Problems

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DRAFT
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These notes consist of a collection of in-class labs created for my Multivariable Calculus classes. The problems are "good" for different reasons, depending on the topic of interest.

http://geometerjustin.com/teaching/c3/gp/

Preface

Write Preface here. Philosophy of the problems, book used in class, *etc.*

Instructions

Complete all exercises. You may work in groups or independently, depending on your personal learning taste. Attempt each problem without using a calculator, and only refer to your calculator (physical or online) if directed by the problem, or if absolutely necessary. If you get stuck on a problem, discusI apologize for all of the mistakes in the Good Problems this week. I have updated everyone's grade to 5/5. Is it with your group and/or neighboring groups. If the group is still stuck, then your entire group may ask me for a hint.

Students are expected to be present and working on these Good Problems in class, during the time allotted by the instructor. Any problems that are not completed during class time are expected to be completed outside of class, before the next class meeting.

Contents

1. Vector-Valued Functions

These Good Problems cover material from sections 12.1 – 12.3 of our book. Topics include an introduction to vector functions and their space curves; derivatives and integrals of vector functions; and arc length and curvature of space curves.

1. Consider the vector function $\mathbf{r}(t) = \left\langle \sqrt{4-t^2}, e^{-3t}, \ln(t+1) \right\rangle$.

a.) What is the domain of $\mathbf{r}$?

b.) Evaluate $\lim_{t \to 0} \mathbf{r}(t)$.

2. Evaluate the limit.

$$\lim_{t \to 1} \left(\frac{t^2 - t}{t-1}\,\mathbf{i} + \sqrt{t+8}\,\mathbf{j} + \frac{\sin(\pi t)}{\ln t}\,\mathbf{k} \right)$$

3. Recall that a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is said to be *continuous* at the point $t = a$ if and only if $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$.

We proved in class that if the component functions x, y, and z are each continuous at $t = a$, then $\mathbf{r}$ is continuous at $t = a$. Prove the converse: If $\mathbf{r}$ is continuous at $t = a$, then so are each of x, y, and z.

4. Find a parametrization of the space curve defined by the intersection of the surfaces $z = 4x^2 + y^2$ and $y = x^2$ in $\mathbb{R}^3$.

5. Sketch the curves in $\mathbb{R}^2$ and the surfaces $\mathbb{R}^3$ defined by the vector functions. Indicate the direction of increasing t.

a.) $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$

b.) $\mathbf{r}(t) = \langle t^2, t \rangle$

6. Sketch the space curves determined by the vector functions.

a.) $\mathbf{r}(t) = \langle \sin t, t, \cos t \rangle$

b.) $\mathbf{r}(t) = t^2\,\mathbf{i} + t\,\mathbf{j} - t\,\mathbf{k}$

Bezier curves.

Let P_0, P_1, P_2, and P_3 be points in $\mathbb{R}^3$: $P_i = (x_i, y_i, z_i)$ for $i = 0, 1, 2, 3$. Regard each point P_i as the terminal point of a vector $\mathbf{P}_i$, identifying the ordered triple (x_i, y_i, z_i) with the vector $\langle x_i, y_i, z_i \rangle$ in $\mathbb{R}^3$. The *Bezier curve* defined by these points (equivalently, vectors) is the space curve associated with the vector function,

$$\mathbf{B}(t) = (1-t)^3\,\mathbf{P}_0 + 3(1-t)^2 t\,\mathbf{P}_1 + 3(1-t)t^2\,\mathbf{P}_2 + t^3\,\mathbf{P}_3, \quad 0 \le t \le 1.$$

7. Determine the Bezier curve for the points

$$P_0(0,1,2),\ P_1(0,0,2),\ P_2(2,1,0),\ \text{and } P_3(2,0,0).$$

Use a graphing utility to graph the curve. Identify each of the points and its relationship to the graph.

8. Set up and simplify the integral that represents the length of the Bezier curve you found in problem **7**. Use a graphing utility or CAS to estimate the length of the curve. Round your answer to two decimal places.

9. Let $y = f(x)$ be a twice-differentiable function. Show that the curvature of f is given by

$$\kappa(x) = \frac{|f''(x)|}{\sqrt{1 + (f'(x))^2}^3}.$$

10. Find a formula for the curvature of the curve $y = \tan x$, and use it to calculate the curvature at the point $(\frac{\pi}{4}, 1)$.

11. Two particles travel along the space curves

$$\mathbf{r}_1(t) = \left\langle t, t^2, t^3 \right\rangle \quad \text{and} \quad \mathbf{r}_2(u) = \langle 1 + 2u, 1 + 6u, 1 + 14u \rangle .$$

Do the particles collide? If not, do their paths intersect?

12. Consider the vector function $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), t\rangle$, $0 \leq t \leq 2\pi$. Compute $\dot{\mathbf{r}}(t)$, $\int \mathbf{r}(t)\,dt$, $s(t)$, and $\kappa(t)$.

13. Find an equation of a parabola that has curvature $\kappa = 4$ at the origin.

14. Use your favorite formula for curvature to prove the following statement: *The curvature of a circle of radius a is constant,* $\kappa = \dfrac{1}{a}$.

For this reason, the number $1/\kappa$ is referred to as the *radius of curvature* at each point of a space curve.

2. Physical Applications

These Good Problems cover material related to section 12.3 of our book. Some of this material is not found in our text, but it is extremely important. Please use my notes as a reference. Topics include tangent, normal, and binormal vectors; curvature and torsion; velocity and acceleration; the Frenet-Serret formulas; and some applications of these ideas.

1. Consider the function $f(x) = x^4 - 2x^2$. Find its curvature function κ and the osculating circle to the curve at the origin. Use a graphing utility to graph the curve $y = x^4 - 2x^2$, its curvature function κ, and the osculating circle on the same set of axes.

2. Find **T**, **N**, and **B**, the curvature κ and the torsion τ of the curve $\mathbf{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle$ at the point $(1, 0, 0)$. Also find equations of the normal and osculating planes to the curve at this point.

3. Find the curvature and torsion of the curve at the point $(0, 1, 0)$.

$$\mathbf{r}(t) = \sinh t\,\mathbf{i} + \cosh t\,\mathbf{j} + t\,\mathbf{k}$$

4. Find the velocity, acceleration, and speed of the particle.

$$\mathbf{r}(t) = e^t \langle \cos t, \sin t, t \rangle$$

5. Find the tangential and normal components of the acceleration vector for the curve

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 3t\,\mathbf{k}.$$

6. A particle has position function $\mathbf{r}$. If $\dot{\mathbf{r}}(t) = \mathbf{c} \times \mathbf{r}(t)$ for all t, where $\mathbf{c}$ is a constant vector, describe the path of the particle.

7. Suppose you need to design a smooth transition between parallel straight sections of a railroad track. Existing track along the negative x-axis is to be joined to a track along the line $y = 1$ for $x \geq 1$.

a.) Find a polynomial function P of degree 5 such that the function F defined by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

is continuous and has continuous slope and curvature.

b.) Use a graphing utility to draw the graph of F and sketch it below.

8. The position function of a spaceship is

$$\mathbf{r}(t) = (3 + t)\,\mathbf{i} + (2 + \ln t)\,\mathbf{j} + \left(7 - \frac{4}{t^2 + 1}\right)\,\mathbf{k}$$

and the coordinates of a space station are $(6, 4, 9)$. The captain wants the spaceship to coast into the space station. When should the engines be turned off?

9. A rocket burning its onboard fuel while moving through space has velocity $\mathbf{v}(t)$ and mass $m(t)$ at time t. If the exhaust gases escape with velocity $\mathbf{v}_e$ relative to the rocket, it can be deduced from Newton's Second Law of Motion that

$$m\frac{d\mathbf{v}}{dt} = \frac{dm}{dt}\mathbf{v}_e.$$

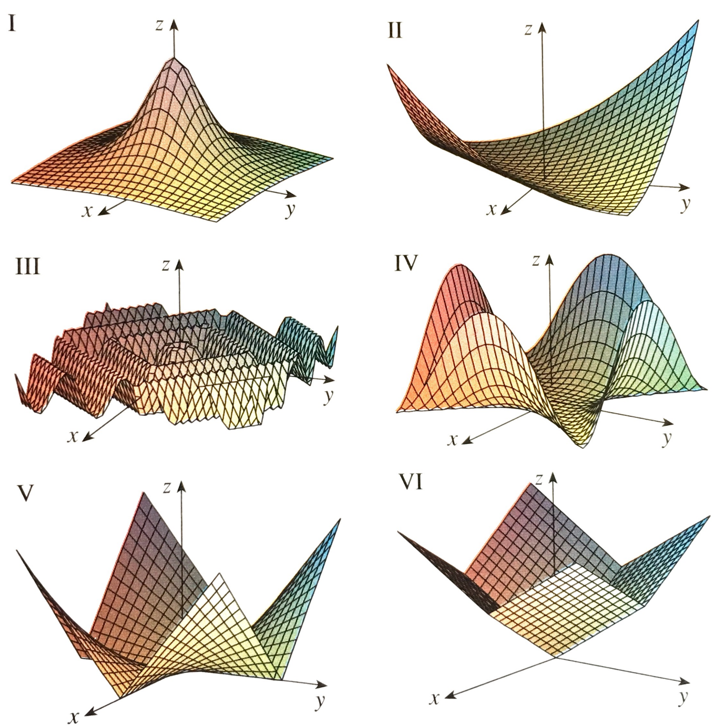
a.) Show that $\mathbf{v}(t) = \mathbf{v}(0) - \ln\left(\frac{m(0)}{m(t)}\right)\mathbf{v}_e$.

b.) For the rocket to accelerate in a straight line from rest to twice the speed of its own exhaust gases, what fraction of its initial mass would the rocket have to burn as fuel?

3. Functions of Several Variables

These Good Problems cover material from sections 13.1 and 13.2 of our book. Topics include functions of two variables, their contour maps, and limits.

1. Match the function with its graph.



______ *a.)* $f(x, y) = |x| + |y|$ ______ *b.)* $f(x, y) = |xy|$

______ *c.)* $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$ ______ *d.)* $f(x, y) = (x^2 - y^2)^2$

______ *e.)* $f(x, y) = (x - y)^2$ ______ *f.)* $f(x, y) = \sin(|x| + |y|)$

2. Find and sketch the domain of the function

$$F(x, y) = \arcsin(x^2 + y^2 - 2).$$

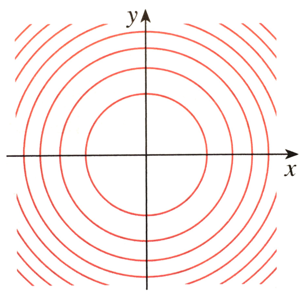
3. Find and sketch the domain of the function

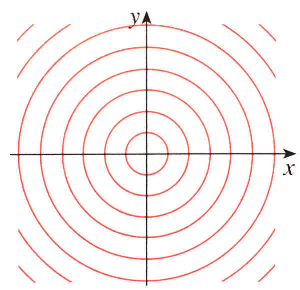
$$G(x, y) = \ln(9 - x^2 - 9y^2).$$

4. Find and sketch the domain of the function

$$f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}.$$

5. Two contour maps are given. One is for a function f whose graph is a cone. The other is for a function g whose graph is a paraboloid. Which is which, and why?

I. 

II. 

6. A thin metal plate, located in the xy-plane, has temperature $T(x, y)$ at the point (x, y). The level curves of T are called *isothermals* because at all points on such a curve the temperature is the same. Sketch some isothermals if the temperature function is given by

$$T(x, y) = \frac{100}{1 + x^2 + 2y^2}.$$

7. Use a computer to investigate the family of functions

$$f(x, y) = e^{cx^2+y^2}.$$

How does the shape of the graph depend on c?

8. Show that the limit does not exist, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$.

9. Use polar coordinates to find the limit. (You may assume the limit exists.)

$$\lim_{(x,y)\to(0,0)} (x^2 + y^2) \ln(x^2 + y^2)$$

10. Use a computer graph of the function to explain why the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 + 3xy + 4y^2}{3x^2 + 5y^2}$$

11. Use the ε-δ definition of limit to prove that the limit exists.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{2x^2 + 2y^2} = 0$$

4. Partial Derivatives, Tangent Planes, Review

These Good Problems cover concepts studied in sections 13.3 and 13.4 of our text, then review concepts from the entire semester as preparation for the semester's first midterm exam.

1. *a.*) Sketch the graph of the function $F(x, y) = \sqrt{1 - x^2 - y^2}$.

b.) Plot the point $p(\frac{1}{4}, \frac{1}{2})$, in the domain of F, and the point $(\frac{1}{4}, \frac{1}{2}, F(p))$ on the surface.

c.) Sketch the tangent lines to the surface in the planes $x = \frac{1}{4}$ and $y = \frac{1}{2}$. Describe the partial derivatives $\partial_y F(p)$ and $\partial_x F(p)$ in terms of these lines.

d.) Sketch the tangent plane to the surface at the point $(\frac{1}{4}, \frac{1}{2}, F(p))$.

2. Calculate the partial derivatives $\partial_x F$ and $\partial_y F$ for the function $F(x, y) = \sqrt{1 - x^2 - y^2}$. Then use them to find an equation of the tangent plane to the surface at the point $(\frac{1}{4}, \frac{1}{2})$.

3. Compute the partial derivatives $\partial_x z$ and $\partial_y z$ for the function $z = \sin(xy) + ye^x$.

4. Find an equation of the tangent plane to the surface

$$x^4 + y^4 + z^4 = 3x^2y^2z^2$$

at the point $(1, 1, 1)$.

5. Find the differential dT of the function $T(u, v, w) = \dfrac{v}{1 + uvw}$.

6. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.

7. Use differentials to approximate the value of f at the point $(5.01, 4.02)$.
$f(x, y) = \sqrt{x^2 - y^2}$

8. Compute all first partial derivatives of the functions. Show enough work.

a.) $f(x, y) = \dfrac{1}{\sqrt{9 - x^2 - y^2}}$

b.) $g(u, v) = ue^{\sin(uv)}$

c.) $F(x, y) = \ln(x^2 + \arctan y)$

d.) $z = \arcsin(x + y)$

e.) $T(x, y) = e^{-x^2-y^2}$

f.) $z = y^5 \sin(\ln x)$

9. Let $\mathbf{r}$ be a smooth vector function in $\mathbb{R}^3$ such that $\ddot{\mathbf{r}}$ exists and $\ddot{\mathbf{r}} \neq 0$. Show that $\dot{\mathbf{T}}(t) \perp \mathbf{T}(t)$ for all values of t in the domain of $\mathbf{r}$.

10. Prove that the curvature of a circle of radius a is constant, $\kappa = \frac{1}{a}$.

11. Let $\mathbf{r}$ be a smooth space curve such that $\ddot{\mathbf{r}}$ exists and $\ddot{\mathbf{r}} \neq 0$. Prove that $\mathbf{B}(t)$ is a unit vector for all t in the domain of $\mathbf{r}$.

12. Find an equation of the osculating circle to the plane curve $x = 1 - y^2$ at the point $(1, 0)$. Show enough work.

13. At what point(s) does the curve $y = x^4 - 6x^2$ have maximum curvature? Show enough work to justify your answer.

14. Consider the space curve $\mathbf{r}(t) = \langle -\cos t, -t, \sin t \rangle$.

a.) Find the arc length function $s = s(t)$ starting at $t = 0$ and in the positive t-direction.

b.) Reparametrize $\mathbf{r}$ with respect to arc length.

c.) Find the curvature $\kappa(s)$ of $\mathbf{r}$.

15. Consider the function $f(x, y) = \ln(9 - x^2 - y^2)$.

a.) State and sketch the domain of f.

b.) Find $\partial_x f$, $\partial_y f$, $\partial_{xy} f$, and $\partial_{yx} f$.

16. Find an vector equation of the tangent line to the surface $f(x, y) = 2x^2 - y^2$ at the point $(1, 1, 1)$ that is parallel to the yz-plane. Show enough work.

17. Find the unit tangent, unit normal, and unit binormal vectors ($\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$) to the curve $\mathbf{r}(t) = \langle \cos t, t, -\sin t \rangle$ at the point $p(0, \frac{\pi}{2}, -1)$.

18. Consider the function $f(x, y) = \sqrt{x^2 + y^2}$ at the point $p(3, 4)$.

a.) Find the linearization $L_p f(x, y)$ of f at p.

b.) Find the differential df_p at p.

c.) Use either the linearization or the differential to estimate the value of $\sqrt{3.01^2 + 3.99^2}$. Leave your answer as a reduced fraction.

19. Find an equation of the normal plane to the space curve $\mathbf{r}(t) = \langle e^t, e^t \cos t, e^t \sin t\rangle$ at the point $p(1, 1, 0)$.

20. Find the tangential and normal components of acceleration for the space curve $\mathbf{r}(t) = 3\cos t\,\mathbf{i} - 4\sin t\,\mathbf{j} + 359\,\mathbf{k}$.

Some Comments

The review portion of these Good Problems is not meant the be comprehensive. You should also study past Good Problems and Recommended Exercises.

The exam will be structured as follows. There will be 5 True/False questions and 5 "Fill in the Blank" questions, each worth 1 point each. Then there will be 14 Multiple Choice questions worth 5 points each. Finally, there will be 5 Short Answer questions worth 5 points each, of which you will choose to complete 4. The Multiple Choice questions are all or nothing (no partial credit), but partial credit will be possible on the Short Answer questions.

You will *not* be allowed to use a calculator or any other electronic device on the exam. You will be allowed to use a single 3×5 in^2 note card of your own hand-written notes. If the note card is too big, or if the notes are not written by hand, then you will not be allowed to use the note card on the exam.

You'll also need to know...

Definitions!

I won't ask you to state any definitions word-for-word, but I will expect you to know them. Definitions are the most important part of this course. If we don't know what the terms mean, then there is no chance we can properly apply the terms to solve problems.

5. Chain Rule, Directional Derivatives

These Good Problems cover material from sections 13.8 and 13.5 of our book. Topics include the chain rule, the gradient, and directional derivatives.

1. Suppose $f = f(x, y)$, $x = x(u, v, w)$, $y = y(u, v, w)$, and $u = u(t)$, $v = v(t)$, $w = w(t)$. Write out $\frac{df}{dt}$ in Leibniz notation.

2. Let $w = xy + yz + zx$ where $x = r\cos\theta$, $y = r\sin\theta$, and $z = r\theta$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ at $(r, \theta) = (2, \frac{\pi}{2})$.

3. Use the formulas for implicit differentiation derived in the lecture to compute $\partial z/\partial x$ and $\partial z/\partial y$ for $x^2 + 2y^2 + 3z^2 = 1$.

4. Use the formulas for implicit differentiation derived in the lecture to compute $\partial z/\partial x$ and $\partial z/\partial y$ for $yz + x \ln y = z^2$.

5. A function is called *homogeneous of degree* n if it satisfies the equation $f(tx, ty) = t^n f(x, y)$ for all t, where n is a positive integer and f has continuous second-order partial derivatives.

a.) Verify that $f(x, y) = x^2y + 2xy^2 + 5y^3$ is homogeneous of degree 3.

b.) Show that if f is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y).$$

[Hint: Use the Chain Rule to differentiate $f(tx, ty)$ with respect to t.]

6. Find the gradient ∇f, evaluate $\nabla f(p)$, and find the rate of change of f at p in the direction of the vector $\mathbf{u}$.

$$f(x,y) = \frac{y^2}{x}, \qquad p(1,2), \qquad \mathbf{u} = \frac{1}{3}(2\,\mathbf{i} + \sqrt{5}\,\mathbf{j})$$

7. Find the directional derivative of the function $f(x,y) = e^x \sin y$ at the point $p(0, \frac{\pi}{3})$ in the direction of the vector $\mathbf{v} = \langle -6, 8 \rangle$.

8. Find the directional derivative of the function $f(x, y) = \sqrt{xy}$ at the point $p(2, 8)$ in the direction of the point $q(5, 4)$.

9. Find the maximum rate of change of $f(x, y) = \sin(xy)$ at the point $(1, 0)$ and the direction in which it occurs.

10. Find all points at which the direction of fastest change of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.

11. Suppose that over a certain region of space the electrical potential V is given by $V(x, y, z) = 5x^2 - 3xy + xyz$. (*a.*) Find the rate of change of the potential at $p(3, 4, 5)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$. (*b.*) In which direction does V change the most rapidly at p? (*c.*) What is the maximum rate of change at p?

12. The *second directional derivative* of $f(x, y)$ is

$$D^2_{\mathbf{u}} f(x, y) = D_{\mathbf{u}} \left[D_{\mathbf{u}} f(x, y) \right].$$

If $f(x, y) = x^3 + 5x^2y + y^3$ and $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$, calculate $D^2_{\mathbf{u}} f(2, 1)$.

13. Let f be a smooth function at a point p, $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$, $\mathbf{v} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$, and suppose $D_{\mathbf{u}} f(p) = 2$ and $D_{\mathbf{v}} f(p) = 3$. Find $\nabla f(p)$.

6. Optimization and Lagrange Multipliers

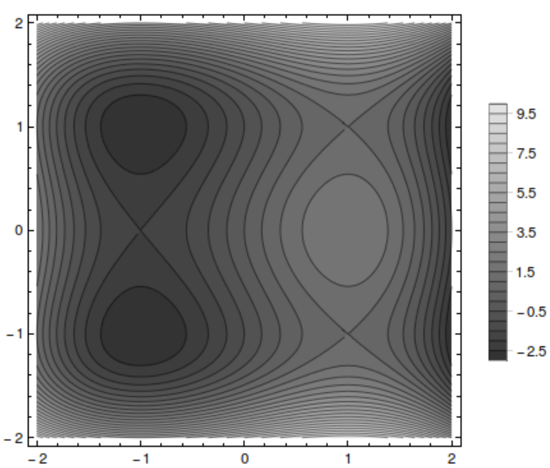
These Good Problems cover material from sections 13.6 and 13.7 of our book.

1. Suppose $(1,1)$ is a critical point of a function f with continuous second derivatives. In each case below, what can you say about f?

a.) $\partial_{xx} f(1,1) = 4$, $\partial_{xy} f(1,1) = 1$, and $\partial_{yy} f(1,1) = 2$

b.) $\partial_{xx} f(1,1) = 4$, $\partial_{xy} f(1,1) = 3$, and $\partial_{yy} f(1,1) = 2$

2. Use the level curves in the figure to predict the location of the critical points of $f(x,y) = 3x - x^3 - 2y^2 + y^4$ and whether f has a saddle point or a local minimum or maximum at each of those points. Use a graphing utility to plot the graph of the function, and compare with the contour map.



3. For functions of one variable it is impossible for a continuous function to have two local maxima and no local minimum. But for functions of two variables such functions do exist. Show that the function

$$f(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2$$

has only two critical points, but has local maxima at both points. Then use a graphing utility to graph the function on a domain that shows both points to see how this is possible.

4. If a function of one variable is continuous on an interval and has only one critical point, then a local maximum must be an absolute maximum. But this is not true for functions of two variables. Show that the function

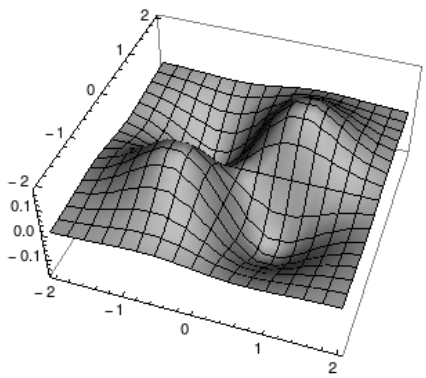
$$f(x, y) = 3xe^y - x^3 - e^{3y}$$

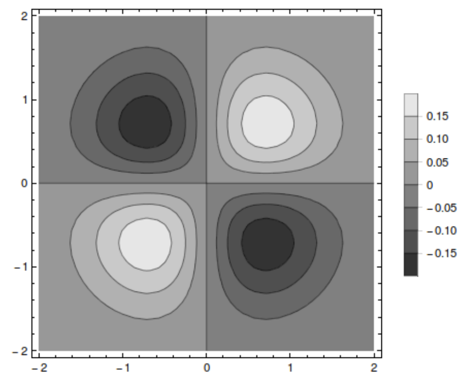
has exactly one critical point, and that f has a local maximum there that is not an absolute maximum. Then use a graphing utility to graph the function on a domain that shows how this is possible.

5. Use the graph and contour plot to estimate the local maxima, local minima, and saddle points (if they exist) of the function

$$f(x, y) = xye^{-x^2-y^2}$$

then use Calculus to find these values precisely.





6. Find the dimensions of the rectangular box with largest volume if the total surface area is given to be 64 cm^2.

7. Find three positive numbers x, y, z whose sum is 100 such that xy^2z^3 is a maximum.

8. Use Lagrange multipliers to find the maximum and minimum values (if they exist) of the function

$$f(x, y) = x^2 + y^2$$

subject to the constraint $xy = 1$.

9. Find the extreme values of the function

$$f(x, y) = 2x^2 + 3y^2 - 4x - 5$$

on the region $x^2 + y^2 \leq 16$.

[Hint: Use the gradient method on the inside and Lagrange multipliers on the boundary.]

10. Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.

7. Double Integrals

This week's Good Problems cover material from sections 14.1 and 14.2 of our book. Topics include the integration of functions of two variables over regions in the xy-plane.

1. Evaluate the double integrals by first identifying them as the volume of solids.

a.) $\displaystyle\iint_R 3\,dA, \quad R = \{(x,y) \mid -2 \leq x \leq 2,\ 1 \leq y \leq 6\}$

b.) $\displaystyle\iint_R (5-x)\,dA, \quad R = \{(x,y) \mid 0 \leq x \leq 5,\ 0 \leq y \leq 3\}$

2. The integral $\iint_R \sqrt{9-y^2}\, dA$, where $R = [0,4] \times [0,2]$ represents the volume of a solid. Sketch the solid.

3. Find the volume of the solid that lies under the hyperbolic paraboloid $z = 4 + x^2 - y^2$ and above the square $R = [-1,1] \times [0,2]$.

4. Calculate the iterated integrals.

a.) $\displaystyle\int_1^3\int_0^1 (1+4xy)\,dx\,dy$

b.) $\displaystyle\int_0^2\int_0^{\frac{\pi}{2}} x\ \sin y\,dy\,dx$

c.) $\displaystyle\int_0^1\int_1^2 \frac{xe^x}{y}\,dy\,dx$

d.) $\displaystyle\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dy\,dx$

5. Calculate the double integrals.

a.) $\displaystyle\iint_R \frac{xy^2}{x^2+1}\,dA, \quad R = \{(x,y) \mid 0 \le x \le 1,\ -3 \le y \le 3\}$

b.) $\displaystyle\iint_R x\sin(x+y)\,dA, \quad R = \left[0, \tfrac{\pi}{6}\right] \times \left[0, \tfrac{\pi}{3}\right]$

c.) $\displaystyle\iint_R \frac{x}{1+xy}\,dA, \quad R = [0,1] \times [0,1]$

6. The *average value* of a function f over a rectangle R is defined to be

$$f_{avg} = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\, dA.$$

Find the average value of $f(x,y) = x^2y$ where R has vertices $(-1,0)$, $(-1,5)$, $(1,5)$, and $(1,0)$.

7. In what way are Fubini's Theorem and Clairaut's Theorem similar? If $f(x,y)$ is continuous on the rectangle $[a,b] \times [c,d]$ and

$$g(x,y) = \int_a^x \int_c^y f(u,v)\, du\, dv$$

for $a < x < b$ and $c < y < d$. Show that $\partial_x \partial_y g = \partial_y \partial_x g = f(x,y)$.

8. Evaluate the double integral

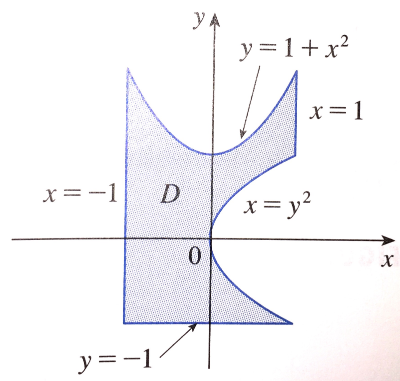
$$\iint_D xy^2\, dA$$

where D is the region enclosed by the curves $x = 0$ and $x = \sqrt{1 - y^2}$.

9. Find the volume of the solid bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, and $z = 0$ in the first octant.

10. Express D as a union of a type I and type II region, then evaluate the integral

$$\iint_D xy\,dA$$



11. Evaluate the integral by reversing the order of integration.

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy$$

8. Change of Coordinates in Double Integrals

This week's Good Problems cover material from sections 14.8 and 14.3 of our book. Topics include change of variables in double integrals, Jacobians, linear transformations, and polar coordinates.

1. Let $S = \{(u, v) \mid 0 \leq u \leq 1, \quad 0 \leq v \leq 1\}$ be a square in the uv-plane. Sketch the image of S under the transformation

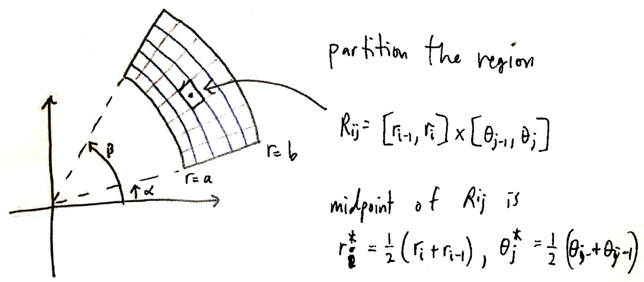
$$T : (u, v) \mapsto (x, y) = \big(v, u(1 + v^2)\big) .$$

Compute the Jacobian of T.

2. A **polar rectangle** is a portion of an annulus defined by

$$R = \{(r, \theta) \mid a \le r \le b, \quad \alpha \le \theta \le \beta\}.$$

Suppose we partition the region R into $m \times n$ sub-regions as indicated in the figure.



Use elementary geometry techniques to show that the infinitesimal change in area ΔA_{ij} of the sub-region R_{ij} is

$$\Delta A_{ij} = r_i^* \Delta r_i \Delta \theta_j.$$

Use this information to write the Riemann sum definition of the double integral

$$\iint_R f(r, \theta)\, dA.$$

What happens to ΔA_{ij} in the limit?

3. Evaluate the integral by making an appropriate change of coordinates.

$$\iint_R (x+y)e^{x^2-y^2}\, dA$$

where R is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$.

Use this page to show your work for GP 8.3

4. Evaluate the integral by making an appropriate change of coordinates.

$$\iint_R \cos\left(\frac{y-x}{y+x}\right)\, dA$$

where R is the trapezoid with vertices $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$.

Use this page to show your work for GP 8.4

5. Evaluate the integral by making an appropriate change of coordinates.

$$\iint_R \sin(9x^2 + 4y^2)\, dA$$

where R is the region in the first quadrant bounded by the ellipse $9x^2 + 4y^2 = 1$.

Use this page to show your work for GP 8.5

6. Use polar coordinates to combine the sum

$$\int_{\frac{1}{\sqrt{2}}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dydx + \int_{1}^{\sqrt{2}}\int_{0}^{x} xy\,dydx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\,dydx$$

into a single integral, then evaluate the integral.

7. Let f be continuous on $[0,1]$ and let R be the triangular region with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Show that

$$\iint_R f(x+y)\,dA = \int_0^1 u\,f(u)\,du.$$

8. Compute

$$\iint_R \frac{1}{1+(x+y)^2}\,dA$$

where R is the triangular region with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

9. (a.) Define the improper integral

$$I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)}\, dA$$

where D_a is the disk with radius a centered at the origin. Show that $I = \pi$.

(b.) Equivalently, we may define

$$I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \lim_{a\to\infty} \iint_{S_a} e^{-(x^2+y^2)}\, dA.$$

where S_a is the square with vertices $(\pm a, \pm a)$. Show that

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy.$$

(c.) Deduce that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

(d.) Use the change of coordinates $u = \sqrt{2}\, x$ to show that

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\,\pi}\,.$$

This is a fundamental result in statistics and probability (area under a normal curve).

Use this page to show your work for GP 8.9

9. Triple Integrals

This week's Good Problems cover material from sections 14.5, 14.6, and 14.7 of our book. Topics include change of variables in triple integrals in rectangular, cylindrical, and spherical coordinates.

1. Work out the details to show that $dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$ in spherical coordinates (*i.e.*, compute the Jacobian).

2. Sketch the domain of the triple integral, then use an appropriate coordinate system to evaluate the integral.

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2+y^2+z^2} \, dz \, dx \, dy$$

3. Sketch the solid whose volume is given by the iterated integral, then evaluate the integral.

$$\int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\, dz\, dx$$

4. Sketch the domain, then write five other iterated integrals that are equal to the given iterated integral.

$$\int_0^1 \int_0^{x^2} \int_0^{y} f(x,y,z)\; dz\, dy\, dx$$

5. Evaluate the integral by using an appropriate coordinate system.

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\;dz\,dy\,dx$$

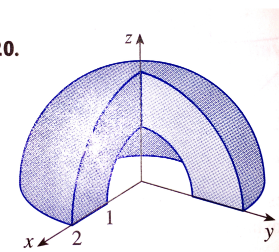
6. Sketch the solid whose volume is given by the integral, then evaluate the integral.

$$\int_{0}^{\frac{\pi}{2}}\int_{0}^{2}\int_{0}^{9-r^2} r\,dz\,dr\,d\theta$$

7. The surfaces $\rho = 1 + \frac{1}{5}\sin m\theta \sin n\varphi$ have been used as models for tumors. Use a computer algebra system [CAS] (*e.g.*, Wolfram—Alpha) to graph the "bumpy sphere" with $m = 6$ and $n = 5$. Then use the CAS to compute the volume it encloses.

8. Set up a triple integral in spherical coordinates to compute the volume of the region shown, then evaluate the integral.

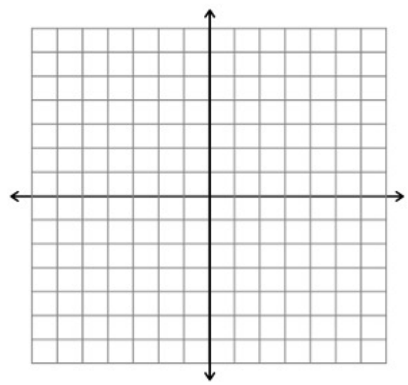
20.



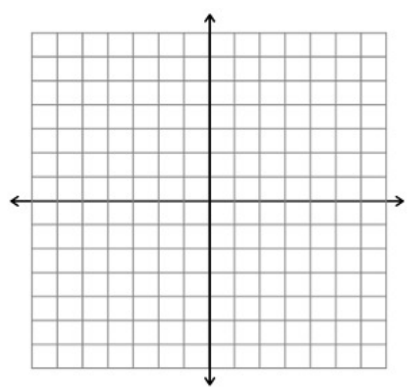
10. Vector Fields and Path Integrals

This week's Good Problems cover material from sections 15.1 and 15.3 of our book. Topics include vector fields and path integrals (known as "line" integrals in the book and in other areas of math, but this name is deceiving).

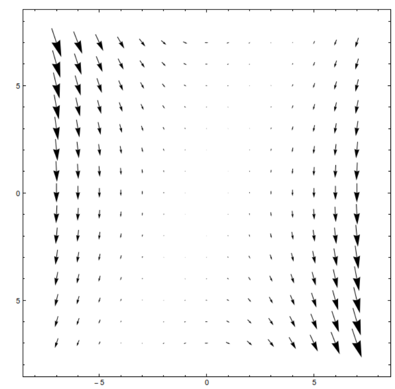
1. Sketch (some of) the vector field $\mathbf{F}(x, y) = \frac{1}{2}(\mathbf{i} + \mathbf{j})$.



2. Sketch (some of) the vector field $\mathbf{F}(x, y) = \left\langle \dfrac{y}{\sqrt{x^2 + y^2}}, \dfrac{-x}{\sqrt{x^2 + y^2}} \right\rangle$.



3. Consider the vector field $\mathbf{F}(x, y) = (y^2 - 2xy)\mathbf{i} + (3xy - 6x^2)\mathbf{j}$.



Describe the appearance by finding the set of points (x, y) satisfying $\mathbf{F}(x, y) = \mathbf{0}$.

4. Let $\mathbf{x} = \langle x, y \rangle$, $r = \|\mathbf{x}\|$, and $\mathbf{F}(\mathbf{x}) = (r^2 - 2r)\mathbf{x}$. Use a `CAS` (such as Wolfram|Alpha) to plot the vector field on various domains, then describe its appearance by finding the points where $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.

5. A particle moves in a velocity field $\mathbf{V}(x, y) = \langle x^2, x + y^2\rangle$. If it is at position $(2, 1)$ at time $t = 3$, estimate its location at time $t = 3.01$.

[Hint: Use the velocity vector at $t = 3$ to write a linear approximation of the position function.]

6. The **flow lines** (or **stream lines**) of a vector field $\mathbf{F}$ are the paths followed by a particle whose velocity field is the given vector field: $\gamma(t) = \langle x(t), y(t)\rangle$ such that $\dot{\gamma}(t) = \mathbf{F}(x(t), y(t))$. Thus a vector field is tangent to its flow lines.

Find the flow line of the vector field $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$ passing through the point $(1, 1)$.

7. Evaluate the path integral $\int_C xy^4\, ds$ where C is the right half of the circle $x^2 + y^2 = 16$.

8. Evaluate $\int_C 2x + 9z\, ds$ where C is the path parametrized by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$.

9. Evaluate the path integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = \langle z, y, -x \rangle$ and $\mathbf{r}(t) = \langle t, \sin t, \cos t \rangle$, $0 \leq t \leq \pi$.

10. Use a `CAS` to plot the vector field $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + xy\,\mathbf{j}$ and the curve $C : x^2 + y^2 = 4$ traversed clockwise from $(2, 0)$ to $(-2, 0)$ on the same set of axes.

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

11. Path Integrals and Green's Theorem

This week's Good Problems cover material from sections 15.4 and 15.6 of our book. Topics include the Fundamental Theorem for path integrals and Green's Theorem.

1. Prove the fundamental theorem for path integrals:

Let $\mathbf{F}$ *be a conservative vector field along a smooth path* C*, and* $\mathbf{r}(t)$*,* $a \le t \le b$*, any parametrization. If* $\mathbf{F}$ *is continuous along* C*, then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

2. Use Clairaut's Theorem to prove:

If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ *is a conservative vector field, where* P *and* Q *have continuous first-order partial derivatives on a domain* D, *then*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout D.

3. Use Green's Theorem to prove:

Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ *be a vector field on an open, simply connected region* D. *Suppose that* P *and* Q *have continuous first-order partial derivatives and*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout D*; then* $\mathbf{F}$ *is conservative.*

Conservation of Energy

Newton's Second Law says that $\mathbf{F} = m\mathbf{a}$. Remembering that $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$ and $\mathbf{a}(t) = \ddot{\mathbf{r}}(t)$, we get a second order differential equation (or SODE):

$$\mathbf{F}(\mathbf{r}(t)) = m\ddot{\mathbf{r}}(t).$$

Let C be a path defined by the parametrized curve $\mathbf{r}(t)$, $t \in [a, b]$. Furthermore, let $\mathbf{r}(a) = A$ and $\mathbf{r}(b) = B$.

4. Use this and the fact that kinetic energy is given by $K(\mathbf{r}(t)) = \frac{1}{2}m\left\|\mathbf{v}(t)\right\|^2$ to show that

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = K(B) - K(A).$$

Suppose that $\mathbf{F}$ is a conservative vector field; *i.e.,* $\mathbf{F} = \nabla f$. The *potential energy* of an object at a point (x, y, z) is defined to be $P(x, y, z) = -f(x, y, z)$ (hence the name *potential function*). Therefore $\mathbf{F} = -\nabla P$.

5. Show that

$$P(A) + K(A) = P(B) + K(B).$$

This is called the *Law of Conservation of Energy.*

6. Find the conservative vector field determined by the potential function $f(x,y) = x\tan(x) - x^2\sec(y)$.

7. Determine whether the vector field is conservative. If so, find its potential function.

$$\mathbf{F}(x,y) = \left\langle \frac{2}{2x+3y}, \frac{3}{2x+3y} + 3y^2 \right\rangle$$

8. Compute

$$\int_C \left(3y + 12x \arctan(x^2 - 9)\right)\, dx - \left(\frac{9 \arcsin y}{\sqrt{4 - y^2}} - 6x\right)\, dy$$

where C is the unit circle $x^2 + y^2 = 1$ traversed counter-clockwise.

9. Let $\mathbf{F}(x, y) = \langle x^2 + y^2, x^2 - y^2 \rangle$. Compute

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

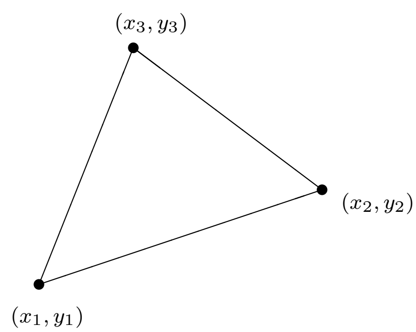
where C is the square with vertices $(-1, -1)$, $(1, -1)$, $(1, 1)$, and $(-1, 1)$, oriented in this order.

One application of Green's Theorem that we talked about in class was that it can be used to calculate the area of an enclosed region by computing a path integral around the boundary of the region. There are many different formulas that can be used to do this, but one of them is

$$\text{Area}(D) = \iint_D 1\, dA = \int_{\partial D} x\, dy.$$

This formula can be used to show (**do it!**) that the area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3), ordered in positive (counter-clockwise) orientation, is given by

$$\text{Area}(\Delta) = \frac{1}{2}\left[x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3\right].$$



This is called the *Shoelace Formula* for area, and can be extended to polygons with n vertices *mutatis mutandis*.

10. Use the shoelace formula to find the area of the triangle with vertices $(0, 0)$, $(2, 5)$, and $(6, 2)$.