1. Vector-Valued Functions

These Good Problems cover material from sections 12.1 – 12.3 of our book. Topics include an introduction to vector functions and their space curves; derivatives and integrals of vector functions; and arc length and curvature of space curves.

1. Consider the vector function $\mathbf{r}(t) = \left\langle \sqrt{4 - t^2}, e^{-3t}, \ln(t+1) \right\rangle$.

a.) What is the domain of $\mathbf{r}$?

b.) Evaluate $\lim_{t \to 0} \mathbf{r}(t)$.

2. Evaluate the limit.

$$\lim_{t \to 1} \left(\frac{t^2 - t}{t - 1}\,\mathbf{i} + \sqrt{t+8}\,\mathbf{j} + \frac{\sin(\pi t)}{\ln t}\,\mathbf{k} \right)$$

3. Recall that a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is said to be *continuous* at the point $t = a$ if and only if $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$.

We proved in class that if the component functions x, y, and z are each continuous at $t = a$, then $\mathbf{r}$ is continuous at $t = a$. Prove the converse: If $\mathbf{r}$ is continuous at $t = a$, then so are each of x, y, and z.

4. Find a parametrization of the space curve defined by the intersection of the surfaces $z = 4x^2 + y^2$ and $y = x^2$ in $\mathbb{R}^3$.

5. Sketch the curves in $\mathbb{R}^2$ and the surfaces $\mathbb{R}^3$ defined by the vector functions. Indicate the direction of increasing t.

a.) $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$

b.) $\mathbf{r}(t) = \langle t^2, t \rangle$

6. Sketch the space curves determined by the vector functions.

a.) $\mathbf{r}(t) = \langle \sin t, t, \cos t \rangle$

b.) $\mathbf{r}(t) = t^2\,\mathbf{i} + t\,\mathbf{j} - t\,\mathbf{k}$

Bezier curves.

Let P_0, P_1, P_2, and P_3 be points in $\mathbb{R}^3$: $P_i = (x_i, y_i, z_i)$ for $i = 0, 1, 2, 3$. Regard each point P_i as the terminal point of a vector $\mathbf{P}_i$, identifying the ordered triple (x_i, y_i, z_i) with the vector $\langle x_i, y_i, z_i \rangle$ in $\mathbb{R}^3$. The *Bezier curve* defined by these points (equivalently, vectors) is the space curve associated with the vector function,

$$\mathbf{B}(t) = (1-t)^3\,\mathbf{P}_0 + 3(1-t)^2 t\,\mathbf{P}_1 + 3(1-t)t^2\,\mathbf{P}_2 + t^3\,\mathbf{P}_3, \quad 0 \le t \le 1.$$

7. Determine the Bezier curve for the points

$$P_0(0,1,2),\ P_1(0,0,2),\ P_2(2,1,0),\ \text{and}\ P_3(2,0,0).$$

Use a graphing utility to graph the curve. Identify each of the points and its relationship to the graph.

8. Set up and simplify the integral that represents the length of the Bezier curve you found in problem **7**. Use a graphing utility or CAS to estimate the length of the curve. Round your answer to two decimal places.

9. Let $y = f(x)$ be a twice-differentiable function. Show that the curvature of f is given by

$$\kappa(x) = \frac{|f''(x)|}{\sqrt{1 + (f'(x))^2}^3}.$$

10. Find a formula for the curvature of the curve $y = \tan x$, and use it to calculate the curvature at the point $(\frac{\pi}{4}, 1)$.

11. Two particles travel along the space curves

$$\mathbf{r}_1(t) = \left\langle t, t^2, t^3 \right\rangle \quad \text{and} \quad \mathbf{r}_2(u) = \langle 1 + 2u, 1 + 6u, 1 + 14u \rangle \, .$$

Do the particles collide? If not, do their paths intersect?

12. Consider the vector function $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), t \rangle$, $0 \le t \le 2\pi$. Compute $\dot{\mathbf{r}}(t)$, $\int \mathbf{r}(t)\, dt$, $s(t)$, and $\kappa(t)$.

13. Find an equation of a parabola that has curvature $\kappa = 4$ at the origin.

14. Use your favorite formula for curvature to prove the following statement: *The curvature of a circle of radius a is constant,* $\kappa = \dfrac{1}{a}$.

For this reason, the number $1/\kappa$ is referred to as the *radius of curvature* at each point of a space curve.