2. Physical Applications

These Good Problems cover material related to section 12.3 of our book. Some of this material is not found in our text, but it is extremely important. Please use my notes as a reference. Topics include tangent, normal, and binormal vectors; curvature and torsion; velocity and acceleration; the Frenet-Serret formulas; and some applications of these ideas.

1. Consider the function $f(x) = x^4 - 2x^2$. Find its curvature function κ and the osculating circle to the curve at the origin. Use a graphing utility to graph the curve $y = x^4 - 2x^2$, its curvature function κ, and the osculating circle on the same set of axes.

$\boxed{f(x) = x^4 - 2x^2}$

$f'(x) = 4x^3 - 4x = 4x(x^2-1)$

$f''(x) = 12x^2 - 4 = 4(3x^2-1)$

$$\boxed{\kappa(x) = \frac{4|3x^2-1|}{\left(1+16x^2(x^2-1)^2\right)^{3/2}}}$$

* See PNG of the graph attached.

$\kappa(0) = \frac{4}{1^{3/2}} = 4$

so $r = \frac{1}{\kappa} = \frac{1}{4}$.

Then the osculating circle is given by

$$\boxed{x^2 + (y+1/4)^2 = \frac{1}{16}}$$

2. Find **T**, **N**, and **B**, the curvature κ and the torsion τ of the curve $\mathbf{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle$ at the point $(1, 0, 0)$. Also find equations of the normal and osculating planes to the curve at this point.

$\vec{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle$ $\qquad \vec{r}(0) = \langle 1, 0, 0 \rangle$

$\dot{\vec{r}}(t) = \langle -\sin t, \cos t, -\tan t \rangle \Rightarrow$ $\qquad \dot{\vec{r}}(0) = \langle 0, 1, 0 \rangle$

$\ddot{\vec{r}}(t) = \langle -\cos t, -\sin t, -\sec^2 t \rangle$ $\qquad \ddot{\vec{r}}(0) = \langle -1, 0, -1 \rangle$

$\dddot{\vec{r}}(t) = \langle \sin t, -\cos t, -2\sec^2 t \tan t \rangle$ $\qquad \dddot{\vec{r}}(0) = \langle 0, -1, 0 \rangle$

$\cos(0) = 1, \quad \sec(0) = 1, \quad \sin(0) = 0, \quad \tan(0) = 0$

$$\vec{T}(t) = \frac{\langle -\sin t, \cos t, -\tan t \rangle}{\sqrt{\sin^2 t + \cos^2 t + \tan^2 t}} = \frac{\langle -\sin t, \cos t, -\tan t \rangle}{\sqrt{1 + \tan^2 t}} = \frac{\langle -\sin t, \cos t, -\tan t \rangle}{\sqrt{\sec^2 t}}$$

$$= \langle -\sin t \cos t, \cos^2 t, -\sin t \rangle$$

$$\boxed{\vec{T}(0) = \langle 0, 1, 0 \rangle}$$

$$\dot{\vec{T}}(t) = \langle -\cos^2 t + \sin^2 t, -2\sin t \cos t, -\cos t \rangle$$

$$\dot{\vec{T}}(0) = \langle -1, 0, -1 \rangle \Rightarrow \boxed{\vec{N}(0) = \left\langle \frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right\rangle}$$

$$\boxed{\vec{B}(0) = \vec{T} \times \vec{N}(0) = \left\langle \frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle}$$

$$\kappa(0) = \frac{\| \dot{\vec{r}}(0) \times \ddot{\vec{r}}(0) \|}{\| \dot{\vec{r}}(0) \|^3} = \frac{\| \langle -1, 0, +1 \rangle \|}{1^3} = \frac{\sqrt{2}}{1} = \boxed{\sqrt{2} = \kappa(0)}$$

$$\tau(0) = \frac{(\dot{\vec{r}} \times \ddot{\vec{r}}) \cdot \dddot{\vec{r}}}{\| \dot{\vec{r}} \times \ddot{\vec{r}} \|^2} = \frac{0}{\sqrt{2}^2} = \boxed{0 = \tau(0)}$$

Normal Plane: normal vector is $\vec{T}$: $\langle x-1, y, z \rangle \cdot \langle 0, 1, 0 \rangle = y$. So,

$$\boxed{y = 0}$$

Osculating Plane: normal vector is $\vec{B}$: $\langle x-1, y, z \rangle \cdot \langle -1, 0, 1 \rangle = x + 1 + z$. So,

$$\boxed{x + z = -1}$$

3. Find the curvature and torsion of the curve at the point $(0, 1, 0)$.

$$\mathbf{r}(t) = \sinh t\,\mathbf{i} + \cosh t\,\mathbf{j} + t\,\mathbf{k}$$

$\vec{r}(t) = \langle \sinh t, \cosh t, t \rangle$

$\dot{\vec{r}}(t) = \langle \cosh t, \sinh t, 1 \rangle \Rightarrow \dot{\vec{r}}(0) = \langle 1, 0, 1 \rangle$

$\ddot{\vec{r}}(t) = \langle \sinh t, \cosh t, 0 \rangle \quad \ddot{\vec{r}}(0) = \langle 0, 1, 0 \rangle$

$\dddot{\vec{r}}(t) = \langle \cosh t, \sinh t, 0 \rangle \quad \dddot{\vec{r}}(0) = \langle 1, 0, 0 \rangle$

$$\kappa(0) = \frac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|^3} = \frac{\|\langle -1, 0, 1 \rangle\|}{\|\langle 1, 0, 1 \rangle\|^3} = \frac{\sqrt{2}}{(\sqrt{2})^3} = \boxed{\frac{1}{2} = \kappa(0)}$$

$$\tau(0) = \frac{(\dot{\vec{r}} \times \ddot{\vec{r}}) \cdot \dddot{\vec{r}}}{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|^2} = \frac{\langle -1, 0, 1 \rangle \cdot \langle 1, 0, 0 \rangle}{\|\langle -1, 0, 1 \rangle\|^2} = \frac{-1}{\sqrt{2}^2} = \boxed{\frac{-1}{2} = \tau(0)}$$

4. Find the velocity, acceleration, and speed of the particle.

$$\mathbf{r}(t) = e^t \langle \cos t, \sin t, t \rangle$$

$\vec{r}(t) = \langle e^t \cos t, e^t \sin t, te^t \rangle$

$\dot{\vec{r}}(t) = \boxed{\langle e^t \cos t - e^t \sin t,\ e^t \cos t + e^t \sin t,\ e^t + te^t \rangle = \vec{v}(t)}$

$\ddot{\vec{r}}(t) = \langle \cancel{e^t \cos t} - e^t \sin t - \cancel{e^t \cos t} - e^t \sin t,\ e^t \cos t - \cancel{e^t \sin t} + \cancel{e^t \sin t} + e^t \cos t,\ 2e^t + te^t \rangle$

$\boxed{\vec{a}(t) = \langle -2e^t \sin t,\ 2e^t \cos t,\ (2+t)e^t \rangle}$

$v(t) = \|\vec{v}(t)\| = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2 + (1+t)^2 e^{2t}}$

$= e^t \sqrt{\cos^2 t - \cancel{2\sin t \cos t} + \sin^2 t + \cos^2 t + \cancel{2\sin t \cos t} + \sin^2 t + 1 + 2t + t^2}$

$= e^t \sqrt{2\cos^2 t + 2\sin^2 t + 1 + 2t + t^2}$

$\boxed{v(t) = e^t \sqrt{3 + 2t + t^2}}$

5. Find the tangential and normal components of the acceleration vector for the curve

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 3t\,\mathbf{k}.$$

$a_T = \dot{v}(t)$ $\quad a_N = \kappa v^2(t)$

$\vec{r}(t) = \langle t, t^2, 3t \rangle$

$\dot{\vec{r}}(t) = \langle 1, 2t, 3 \rangle$

$\ddot{\vec{r}}(t) = \langle 0, 2, 0 \rangle$

$v(t) = \sqrt{4t^2 + 10}$

$\dot{v}(t) = \dfrac{8t}{2\sqrt{4t^2+10}} = \dfrac{4t}{\sqrt{4t^2+10}}$

$\kappa(t) = \dfrac{\|\dot{\vec{r}}(t) \times \ddot{\vec{r}}(t)\|}{\|\dot{\vec{r}}(t)\|^3} = \dfrac{\|\langle -6, 0, 2 \rangle\|}{\left(\sqrt{4t^2+10}\right)^3} = \dfrac{\sqrt{40}}{\left(\sqrt{4t^2+10}\right)^3}$

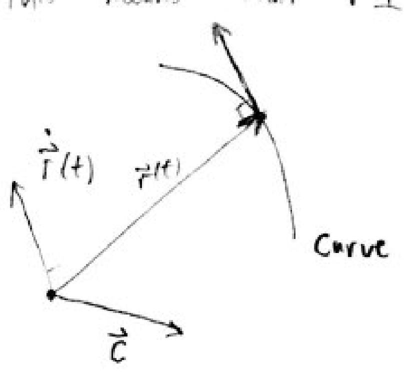
So,

$$\boxed{a_T(t) = \frac{4t}{\sqrt{4t^2+10}}}$$

$$\boxed{a_N(t) = \sqrt{\frac{40}{4t^2+10}}}$$

6. A particle has position function $\mathbf{r}$. If $\dot{\mathbf{r}}(t) = \mathbf{c} \times \mathbf{r}(t)$ for all t, where $\mathbf{c}$ is a constant vector, describe the path of the particle.

Recall that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. Thus, this means that $\dot{\vec{r}} \perp \vec{r}$ for all t.



This means that the curve traced out by the particle is a circle in the plane whose normal vector is $\vec{c}$.

i.e., $\vec{c}$ is the direction of the binormal $\vec{B}$ at each point.

7. Suppose you need to design a smooth transition between parallel straight sections of a railroad track. Existing track along the negative x-axis is to be joined to a track along the line $y = 1$ for $x \geq 1$.

a.) Find a polynomial function P of degree 5 such that the function F defined by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

is continuous and has continuous slope and curvature.

b.) Use a graphing utility to draw the graph of F and sketch it below.

Function continuous: $P(0) = 0$
$P(1) = 1$

Slope continuous: $P'(0) = 0$
$P'(1) = 0$

Curvature continuous: $P''(0) = 0$
$P''(1) = 0$

* See PNG of the graph attached

$P(x) = a_5x^5 + a_4x^4 + a_3x^3 + \cancel{a_2x^2} + \cancel{a_1x} + \cancel{a_0}$

$P'(x) = 5a_5x^4 + 4a_4x^3 + 3a_3x^2 + \cancel{2a_2x} + \cancel{a_1}$

$P''(x) = 20a_5x^3 + 12a_4x^2 + 6a_3x + \cancel{2a_2}$

The constraints $P(0) = P'(0) = P''(0) = 0$ imply that $a_0 = a_1 = a_2 = 0$.

The constraints $P(1) = 1$, $P'(1) = P''(1) = 0$ give a system of 3 equations and three unknowns.

$$\begin{cases} a_5 + a_4 + a_3 = 1 \\ 5a_5 + 4a_4 + 3a_3 = 0 \\ 20a_5 + 12a_4 + 6a_3 = 0 \end{cases}$$

In matrix form,

$$\begin{pmatrix} 1 & 1 & 1 \\ 5 & 4 & 3 \\ 20 & 12 & 6 \end{pmatrix} \begin{pmatrix} a_5 \\ a_4 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

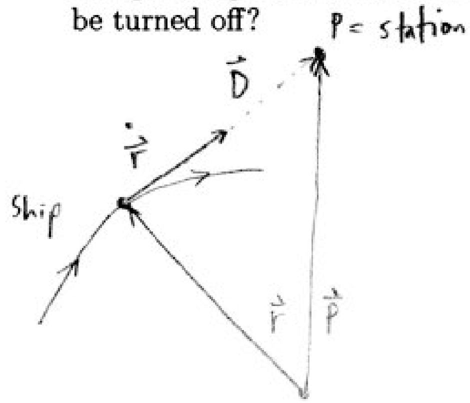
The solution is, $\begin{pmatrix} a_5 \\ a_4 \\ a_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -15 \\ 10 \end{pmatrix}$

So, $\boxed{P(x) = 6x^5 - 15x^4 + 10x^3}$

8. The position function of a spaceship is

$$\mathbf{r}(t) = (3+t)\,\mathbf{i} + (2+\ln t)\,\mathbf{j} + \left(7 - \frac{4}{t^2+1}\right)\mathbf{k}$$

and the coordinates of a space station are $(6, 4, 9)$. The captain wants the spaceship to coast into the space station. When should the engines be turned off?



We're looking for a point on the curve where the velocity is in the direction of the station,

$$\gamma\dot{\vec{r}} = \vec{D} = \vec{P} - \vec{r}$$

where γ is a constant of proportionality.

$$\vec{r}(t) = \left\langle 3+t,\ 2+\ln t,\ 7-\frac{4}{t^2+1}\right\rangle$$

$$\dot{\vec{r}}(t) = \left\langle 1,\ \frac{1}{t},\ \frac{8t}{(t^2+1)^2}\right\rangle \longrightarrow \gamma\dot{\vec{r}} = \left\langle \gamma,\ \frac{\gamma}{t},\ \frac{8\gamma t}{(t^2+1)^2}\right\rangle$$

$$\vec{P}-\vec{r} = \left\langle 6-(3+t),\ 4-(2+\ln t),\ 9-\left(7-\frac{4}{t^2+1}\right)\right\rangle$$

$$= \left\langle 3-t,\ 2-\ln t,\ 2+\frac{4}{t^2+1}\right\rangle = \left\langle 3-t,\ 2-\ln t,\ \frac{2t^2+6}{t^2+1}\right\rangle$$

So, $\boxed{\gamma = 3-t}$

$$\frac{\gamma}{t} = 2-\ln t \Rightarrow \frac{3-t}{t} = 2-\ln t \Rightarrow 3-t = 2t - t\ln t \Rightarrow \boxed{3t - t\ln t - 3 = 0}$$

Solving w/ GeoGebra (or another CAS), $t=1$, $t=16.8$

$$\frac{8\gamma t}{(t^2+1)^2} = \frac{2t^2+6}{t^2+1} \Rightarrow 8(3-t)t = (2t^2+6)(t^2+1) \Rightarrow 24t - 8t^2 = 2t^4 + 8t^2 + 6$$

$$\Rightarrow \boxed{2t^4 + 16t^2 - 24t + 6 = 0}$$ Solving w/ Geogebra: $t=1$, $t=0.32$

So the captain should turn of the engine at $t=1$. Or at the point $(4, 2, 5)$.

9. A rocket burning its onboard fuel while moving through space has velocity $\mathbf{v}(t)$ and mass $m(t)$ at time t. If the exhaust gases escape with velocity $\mathbf{v}_e$ relative to the rocket, it can be deduced from Newton's Second Law of Motion that

$$m\frac{d\mathbf{v}}{dt} = \frac{dm}{dt}\mathbf{v}_e.$$

a.) Show that $\mathbf{v}(t) = \mathbf{v}(0) - \ln\left(\frac{m(0)}{m(t)}\right)\mathbf{v}_e$.

b.) For the rocket to accelerate in a straight line from rest to twice the speed of its own exhaust gases, what fraction of its initial mass would the rocket have to burn as fuel?

a) $\int d\vec{v} = \vec{v}_e \int \frac{m'}{m}\,dt \Rightarrow \vec{v}(t) = \ln(m)\,\vec{v}_e + \vec{c}$

Let $\vec{v}(0) = \vec{v}_0$ and $m(0) = m_0$. Then

$\vec{v}_0 = \ln(m_0)\,\vec{v}_e + \vec{c}$ or $\vec{c} = \vec{v}_0 - \ln(m_0)\vec{v}_e$

Then $\vec{v}(t) = \vec{v}_0 + (\ln(m) - \ln(m_0))\vec{v}_e$

$= \vec{v}_0 + \ln\left(\frac{m}{m_0}\right)\vec{v}_e$

$= \vec{v}_0 - \ln\left(\frac{m_0}{m}\right)\vec{v}_e$ $\square$

b) $\vec{v}_e$ points in the opposite direction of the motion of the rocket, so $\vec{v}(\tau) = -2\vec{v}_e$, and $\vec{v}_0 = 0$. Then

$$+2\vec{v}_e = +\ln\left(\frac{m_0}{m(\tau)}\right)\vec{v}_e \Rightarrow \ln\left(\frac{m_0}{m(\tau)}\right) = 2$$

Then $\frac{m_0}{m(\tau)} = e^2$ or $\frac{m(\tau)}{m_0} = e^{-2}$.

This means that the rocket must burn 13.5% of its initial mass in fuel.