4. Partial Derivatives, Tangent Planes, Review

These Good Problems cover concepts studied in sections 13.3 and 13.4 of our text, then review concepts from the entire semester as preparation for the semester's first midterm exam.

1. *a.*) Sketch the graph of the function $F(x, y) = \sqrt{1 - x^2 - y^2}$.

b.) Plot the point $p(\frac{1}{4}, \frac{1}{2})$, in the domain of F, and the point $(\frac{1}{4}, \frac{1}{2}, F(p))$ on the surface.

c.) Sketch the tangent lines to the surface in the planes $x = \frac{1}{4}$ and $y = \frac{1}{2}$. Describe the partial derivatives $\partial_y F(p)$ and $\partial_x F(p)$ in terms of these lines.

d.) Sketch the tangent plane to the surface at the point $(\frac{1}{4}, \frac{1}{2}, F(p))$.

2. Calculate the partial derivatives $\partial_x F$ and $\partial_y F$ for the function $F(x, y) = \sqrt{1 - x^2 - y^2}$. Then use them to find an equation of the tangent plane to the surface at the point $(\frac{1}{4}, \frac{1}{2})$.

3. Compute the partial derivatives $\partial_x z$ and $\partial_y z$ for the function $z = \sin(xy) + ye^x$.

4. Find an equation of the tangent plane to the surface

$$x^4 + y^4 + z^4 = 3x^2y^2z^2$$

at the point $(1, 1, 1)$.

5. Find the differential dT of the function $T(u, v, w) = \dfrac{v}{1 + uvw}$.

6. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.

7. Use differentials to approximate the value of f at the point $(5.01, 4.02)$.
$f(x, y) = \sqrt{x^2 - y^2}$

8. Compute all first partial derivatives of the functions. Show enough work.

a.) $f(x,y) = \dfrac{1}{\sqrt{9 - x^2 - y^2}}$

b.) $g(u,v) = ue^{\sin(uv)}$

c.) $F(x,y) = \ln(x^2 + \arctan y)$

d.) $z = \arcsin(x+y)$

e.) $T(x,y) = e^{-x^2-y^2}$

f.) $z = y^5 \sin(\ln x)$

9. Let $\mathbf{r}$ be a smooth vector function in $\mathbb{R}^3$ such that $\ddot{\mathbf{r}}$ exists and $\ddot{\mathbf{r}} \neq 0$. Show that $\dot{\mathbf{T}}(t) \perp \mathbf{T}(t)$ for all values of t in the domain of $\mathbf{r}$.

10. Prove that the curvature of a circle of radius a is constant, $\kappa = \frac{1}{a}$.

11. Let $\mathbf{r}$ be a smooth space curve such that $\ddot{\mathbf{r}}$ exists and $\ddot{\mathbf{r}} \neq 0$. Prove that $\mathbf{B}(t)$ is a unit vector for all t in the domain of $\mathbf{r}$

12. Find an equation of the osculating circle to the plan curve $x = 1 - y^2$ at the point $(1, 0)$. Show enough work.

13. At what point(s) does the curve $y = x^4 - 6x^2$ have maximum curvature? Show enough work to justify your answer.

14. Consider the space curve $\mathbf{r}(t) = \langle -\cos t, -t, \sin t \rangle$.

a.) Find the arc length function $s = s(t)$ starting at $t = 0$ and in the positive t-direction.

b.) Reparametrize $\mathbf{r}$ with respect to arc length.

c.) Find the curvature $\kappa(s)$ of $\mathbf{r}$.

15. Consider the function $f(x, y) = \ln(9 - x^2 - y^2)$.

a.) State and sketch the domain of f.

b.) Find $\partial_x f$, $\partial_y f$, $\partial_{xy} f$, and $\partial_{yx} f$.

16. Find an vector equation of the tangent line to the surface $f(x, y) = 2x^2 - y^2$ at the point $(1, 1, 1)$ that is parallel to the yz-plane. Show enough work.

17. Find the unit tangent, unit normal, and unit binormal vectors (**T**, **N**, and **B**) to the curve $\mathbf{r}(t) = \langle \cos t, t, -\sin t \rangle$ at the point $p(0, \frac{\pi}{2}, -1)$.

18. Consider the function $f(x, y) = \sqrt{x^2 + y^2}$ at the point $p(3, 4)$.

a.) Find the linearization $L_p f(x, y)$ of f at p.

b.) Find the differential df_p at p.

c.) Use either the linearization or the differential to estimate the value of $\sqrt{3.01^2 + 3.99^2}$. Leave your answer as a reduced fraction.

19. Find an equation of the normal plane to the space curve $\mathbf{r}(t) = \langle e^t, e^t \cos t, e^t \sin t \rangle$ at the point $p(1, 1, 0)$.

20. Find the tangential and normal components of acceleration for the space curve $\mathbf{r}(t) = 3 \cos t\, \mathbf{i} - 4 \sin t\, \mathbf{j} + 359\, \mathbf{k}$.

Some Comments

The review portion of these Good Problems is not meant the be comprehensive. You should also study past Good Problems and Recommended Exercises.

The exam will be structured as follows. There will be 5 True/False questions and 5 "Fill in the Blank" questions, each worth 1 point each. Then there will be 14 Multiple Choice questions worth 5 points each. Finally, there will be 5 Short Answer questions worth 5 points each, of which you will choose to complete 4. The Multiple Choice questions are all or nothing (no partial credit), but partial credit will be possible on the Short Answer questions.

You will *not* be allowed to use a calculator or any other electronic device on the exam. You will be allowed to use a single 3×5 in^2 note card of your own hand-written notes. If the note card is too big, or if the notes are not written by hand, then you will not be allowed to use the note card on the exam.

You'll also need to know...

Definitions!

I won't ask you to state any definitions word-for-word, but I will expect you to know them. Definitions are the most important part of this course. If we don't know what the terms mean, then there is no chance we can properly apply the terms to solve problems.