6. Optimization and Lagrange Multipliers

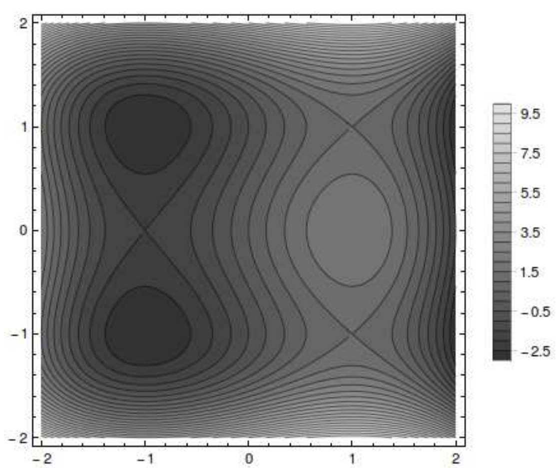
These Good Problems cover material from sections 13.6 and 13.7 of our book.

1. Suppose $(1,1)$ is a critical point of a function f with continuous second derivatives. In each case below, what can you say about f?

a.) $\partial_{xx} f(1,1) = 4$, $\partial_{xy} f(1,1) = 1$, and $\partial_{yy} f(1,1) = 2$

b.) $\partial_{xx} f(1,1) = 4$, $\partial_{xy} f(1,1) = 3$, and $\partial_{yy} f(1,1) = 2$

2. Use the level curves in the figure to predict the location of the critical points of $f(x,y) = 3x - x^3 - 2y^2 + y^4$ and whether f has a saddle point or a local minimum or maximum at each of those points. Use a graphing utility to plot the graph of the function, and compare with the contour map.



3. For functions of one variable it is impossible for a continuous function to have two local maxima and no local minimum. But for functions of two variables such functions do exist. Show that the function

$$f(x,y) = -(x^2-1)^2 - (x^2y-x-1)^2$$

has only two critical points, but has local maxima at both points. Then use a graphing utility to graph the function on a domain that shows both points to see how this is possible.

4. If a function of one variable is continuous on an interval and has only one critical point, then a local maximum must be an absolute maximum. But this is not true for functions of two variables. Show that the function

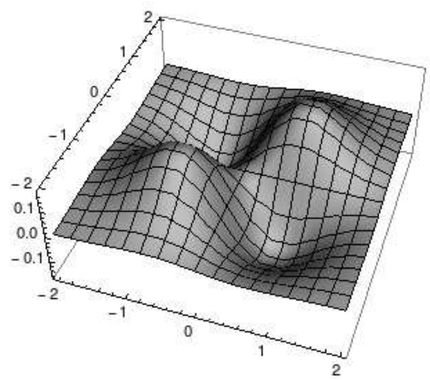
$$f(x, y) = 3xe^y - x^3 - e^{3y}$$

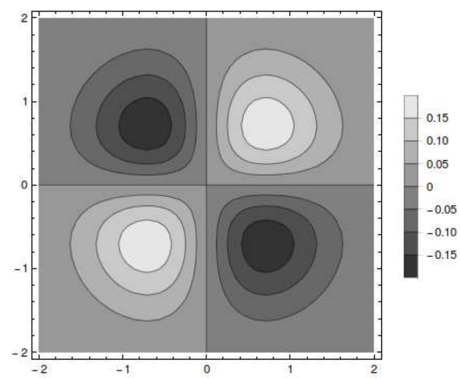
has exactly one critical point, and that f has a local maximum there that is not an absolute maximum. Then use a graphing utility to graph the function on a domain that shows how this is possible.

5. Use the graph and contour plot to estimate the local maxima, local minima, and saddle points (if they exist) of the function

$$f(x, y) = xye^{-x^2-y^2}$$

then use Calculus to find these values precisely.





6. Find the dimensions of the rectangular box with largest volume if the total surface area is given to be 64 cm^2.

7. Find three positive numbers x, y, z whose sum is 100 such that xy^2z^3 is a maximum.

8. Use Lagrange multipliers to find the maximum and minimum values (if they exist) of the function

$$f(x, y) = x^2 + y^2$$

subject to the constraint $xy = 1$.

9. Find the extreme values of the function

$$f(x, y) = 2x^2 + 3y^2 - 4x - 5$$

on the region $x^2 + y^2 \leq 16$.

[Hint: Use the gradient method on the inside and Lagrange multipliers on the boundary.]

10. Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.