9. Triple Integrals

This week's Good Problems cover material from sections 14.5, 14.6, and 14.7 of our book. Topics include change of variables in triple integrals in rectangular, cylindrical, and spherical coordinates.

1. Work out the details to show that $dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$ in spherical coordinates (*i.e.*, compute the Jacobian).

$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{pmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\sin\theta & -\rho\sin\varphi \\ -\rho\sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & 0 \end{pmatrix}$$

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)}\right| = \sin\varphi\cos\theta\left(\rho^2\sin^2\varphi\cos\theta\right) - \sin\varphi\sin\theta\left(-\rho^2\sin^2\varphi\sin\theta\right) +$$
$$+ \cos\varphi\left(\rho^2\cos^2\theta\sin\varphi\cos\varphi + \rho^2\sin^2\theta\sin\varphi\cos\varphi\right)$$
$$= \rho^2\left(\underbrace{\cos^2\theta\sin\varphi\sin^2\varphi + \sin^2\theta\sin\varphi\sin^2\varphi} + \underbrace{\cos^2\theta\sin\varphi\cos^2\varphi + \sin^2\theta\sin\varphi\cos^2\varphi}\right)$$
$$= \rho^2\left(\underbrace{\sin\varphi\sin^2\varphi + \sin\varphi\cos^2\varphi}\right)$$
$$= \rho^2\sin\varphi$$

∎

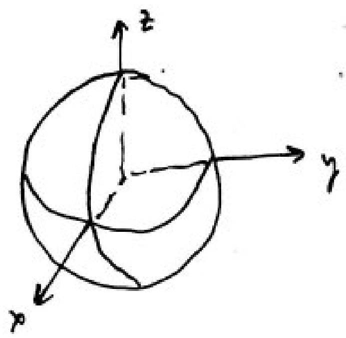
2. Sketch the domain of the triple integral, then use an appropriate coordinate system to evaluate the integral.

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$$

$z: -\sqrt{4-x^2-y^2} \mapsto \sqrt{4-x^2-y^2} \quad : \quad x^2+y^2+z^2=4$

$x: 0 \mapsto \sqrt{4-y^2} \quad : \quad x^2+y^2=4, \; x \geq 0$

$y: -2 \mapsto 2$



∴ "Front" half of a sphere.
In spherical coords:
$\rho: 0 \to 2$
$\varphi: 0 \to \pi$
$\theta: -\frac{\pi}{2} \to \frac{\pi}{2}$

$y^2 = (\rho\sin\varphi\sin\theta)^2 = \rho^2\sin^2\varphi\sin^2\theta$

$\sqrt{x^2+y^2+z^2} = \rho$

$dV = dz\,dx\,dy = \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

The integral becomes,

$$\int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{\pi}\int_0^2 \rho^5\sin^3\varphi\sin^2\theta\,d\rho\,d\varphi\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\theta\,d\theta\int_0^{\pi}\sin^3\varphi\,d\varphi\int_0^2\rho^5\,d\rho$$

$\int_{-\frac{\pi}{2}}^{\pi/2}\sin^2\theta\,d\theta = \frac12\int_{-\frac{\pi}{2}}^{\pi/2} 1-\cos 2\theta\,d\theta = \frac12\left(\theta-\frac12\sin 2\theta\right)\Big|_{-\pi/2}^{\pi/2} = \boxed{\frac12(\pi)}$

$\int_0^{\pi}\sin^3\varphi\,d\varphi = \int_0^{\pi}\sin^2\varphi\sin\varphi\,d\varphi = \int_0^{\pi}(1-\cos^2\varphi)\sin\varphi\,d\varphi = \int_0^{\pi}\sin\varphi\,d\varphi - \int_0^{\pi}\cos^2\varphi\sin\varphi\,d\varphi =$

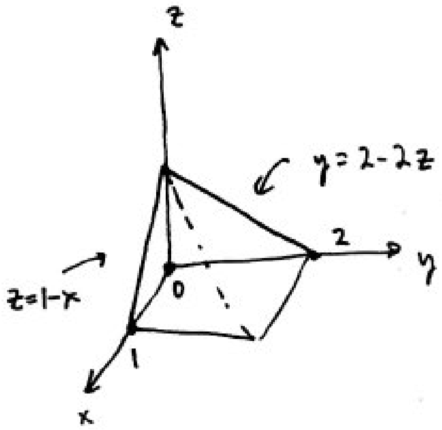
$= (-\cos\varphi)\Big|_0^{\pi} + \int_1^{-1}u^2\,du = (1+1) + \frac{-1}{3} - \frac13 = 2-\frac23 = \boxed{\frac43}$

$\int_0^2\rho^5\,d\rho = \frac16\rho^6\Big|_0^2 = \frac16(64) = \boxed{\frac{32}{3}}$

So the integral equals $\frac12\pi\frac{4}{3}^2\cdot\frac{32}{3} = \boxed{\frac{64}{9}\pi}$ ■

3. Sketch the solid whose volume is given by the iterated integral, then evaluate the integral.

$$\int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\, dz\, dx$$



$$= \int_0^1 \int_0^{1-x} 2-2z \; dz\, dx$$

$$= \int_0^1 \left(2z - z^2\right)\Big|_0^{1-x} dx$$

$$= \int_0^1 2(1-x) - (1-x)^2 \; dx$$

$u = 1-x \qquad u(1) = 0$

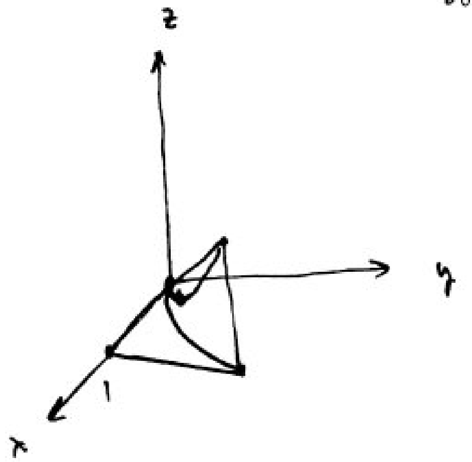
$du = -dx \qquad u(0) = 1$

$$= \int_0^1 2u - u^2 \, du$$

$$= u^2 - \tfrac{1}{3}u^3\Big|_0^1 = 1 - \tfrac{1}{3} = \boxed{\tfrac{2}{3}}$$

4. Sketch the domain, then write five other iterated integrals that are equal to the given iterated integral.

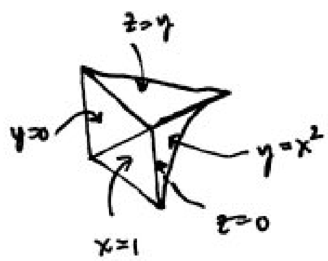
$$1.\quad \int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx$$



$$2.\quad \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f(x,y,z)\, dz\, dx\, dy$$

$$3.\quad \int_0^1 \int_0^y \int_0^{\sqrt{y}} f(x,y,z)\, dx\, dz\, dy$$

$$4.\quad \int_0^1 \int_0^{x^2} \int_0^z f(x,y,z)\, dy\, dz\, dx$$

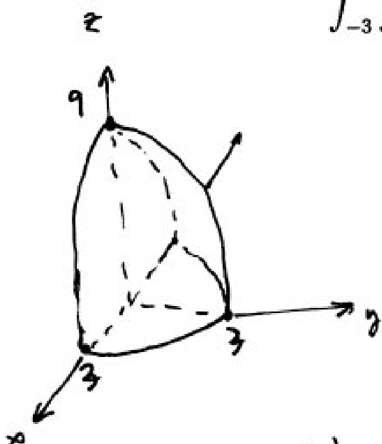


$$5.\quad \int_0^1 \int_0^z \int_{\sqrt{y}}^1 f(x,y,z)\, dx\, dy\, dz$$

$$6.\quad \int_0^1 \int_0^z \int_0^{x^2} f(x,y,z)\, dy\, dx\, dz$$

5. Evaluate the integral by using an appropriate coordinate system.

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$$



In cylindrical coord's: $z: 0 \to 9-r^2$

$r: 0 \to 2$

$\theta: 0 \to \pi$

$\sqrt{x^2+y^2} = r$

$dV = dz\,dy\,dx = r\,dz\,dr\,d\theta$

The integral becomes:

$$\int_0^{\pi}\int_0^{3}\int_0^{9-r^2} r^2\,dz\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{3} 9r^2 - r^4\,dr\,d\theta$$

$$= \int_0^{\pi} d\theta \cdot \left(3r^3 - \tfrac{1}{5}r^5\right)\Big|_0^3$$

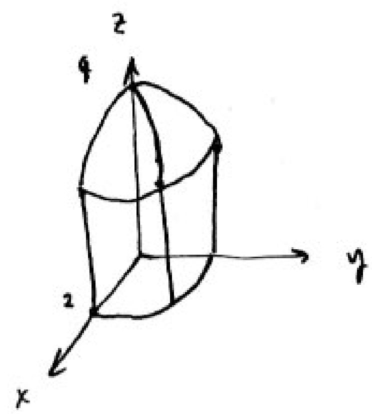
$$= \pi \cdot \left(3\cdot 27 - \frac{243}{6}\right)$$

$$= \pi \cdot \left(\frac{405-243}{5}\right)$$

$$= \boxed{\frac{162\,\pi}{5}}$$

6. Sketch the solid whose volume is given by the integral, then evaluate the integral.

$$\int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{9-r^2} r\, dz\, dr\, d\theta$$



$$= \int_0^{\pi/2} \int_0^2 9r - r^3 \; dr\, d\theta$$

$$= \int_0^{\pi/2} d\theta \cdot \left(\frac{9}{2} r^2 - \frac{1}{4} r^4 \right)\Big|_0^2$$

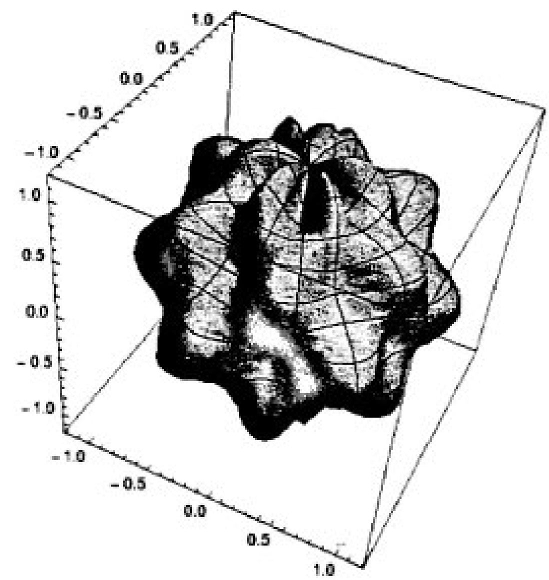
$$= \frac{\pi}{2} \cdot (18 - 4)$$

$$= \frac{14}{2} \pi$$

$$= \boxed{7\pi}$$

7. The surfaces $\rho = 1 + \frac{1}{5}\sin m\theta \sin n\varphi$ have been used as models for tumors. Use a computer algebra system [CAS] (*e.g.*, Wolfram—Alpha) to graph the "bumpy sphere" with $m = 6$ and $n = 5$. Then use the CAS to compute the volume it encloses.

The graph looks like:



Before we can use a CAS to compute the volume, we need to write the triple integral:

$\rho: 0 \mapsto 1+\frac{1}{5}\sin(6\theta)\sin(5\varphi)$

$\varphi: 0 \to \pi$

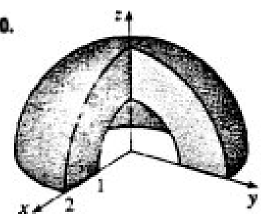
$\theta: 0 \to 2\pi$

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi}\int_0^{1+\frac{1}{5}\sin(6\theta)\sin(5\varphi)} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

Wolfram|Alpha gives $\text{Volume} = \boxed{\frac{136\pi}{99}}$

8. Set up a triple integral in spherical coordinates to compute the volume of the region shown, then evaluate the integral.

20.



Spherical coordinates: $\rho: 1 \to 2$

$\varphi: 0 \to \pi$

$\theta: \frac{\pi}{2} \to 2\pi$

$$\text{Volume} = \int_{\frac{\pi}{2}}^{2\pi} \int_0^{\pi/2} \int_1^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_{\frac{\pi}{2}}^{2\pi} d\theta \cdot \int_0^{\pi/2} \sin\varphi \, d\varphi \cdot \int_1^2 \rho^2 \, d\rho$$

$$= \left(2\pi - \frac{\pi}{2}\right) \cdot \left(-\cos\varphi \Big|_0^{\pi/2}\right) \cdot \left(\frac{1}{3}\rho^3 \Big|_1^2\right)$$

$$= \frac{3\pi}{2}\left(1\right)\left(\frac{8}{3} - \frac{1}{3}\right)$$

$$= \boxed{\frac{7}{2}\pi}$$