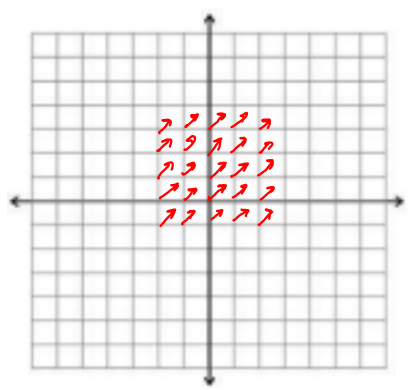
10. Vector Fields and Path Integrals

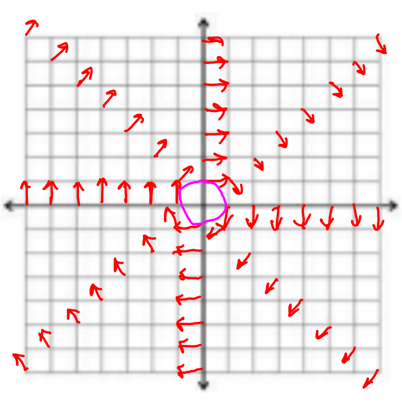
This week's Good Problems cover material from sections 15.1 and 15.3 of our book. Topics include vector fields and path integrals (known as "line" integrals in the book and in other areas of math, but this name is deceiving).

1. Sketch (some of) the vector field $\mathbf{F}(x, y) = \frac{1}{2}(\mathbf{i} + \mathbf{j})$.



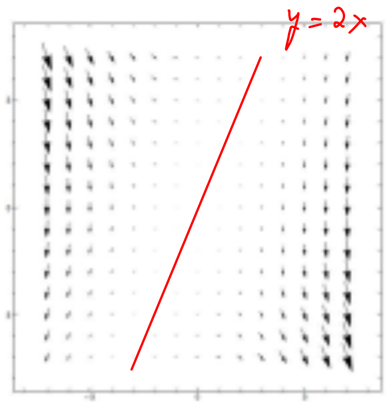
Constant vector field:
the same vector is attached at each point

2. Sketch (some of) the vector field $\mathbf{F}(x, y) = \left\langle \frac{y}{\sqrt{x^2 + y^2}}, \frac{-x}{\sqrt{x^2 + y^2}} \right\rangle$.



Notice that $\|\vec{F}\| = 1$ for all x, y and the vector at each point is tangent to the circle through the point centered at the origin.

3. Consider the vector field $\mathbf{F}(x, y) = (y^2 - 2xy)\mathbf{i} + (3xy - 6x^2)\mathbf{j}$.



Describe the appearance by finding the set of points (x, y) satisfying $\mathbf{F}(x, y) = \mathbf{0}$.

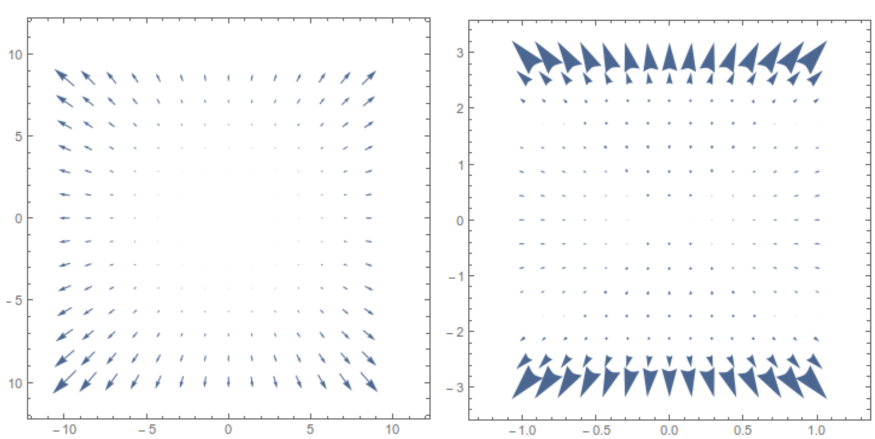
$$\begin{cases} y^2 - 2xy = 0 \\ 3xy - 6x^2 = 0 \end{cases} \rightarrow \begin{cases} y(y-2x) = 0 \\ 3x(y-2x) = 0 \end{cases}$$

$y=0$ and $x=0$ clearly are not solutions,

so

$\boxed{y = 2x}$

4. Let $\mathbf{x} = \langle x, y \rangle$, $r = \|\mathbf{x}\|$, and $\mathbf{F}(\mathbf{x}) = (r^2 - 2r)\mathbf{x}$. Use a CAS (such as Wolfram|Alpha) to plot the vector field on various domains, then describe its appearance by finding the points where $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.



$r^2 - 2r = 0 \Rightarrow r(r-2) = 0$

$r = 0 \rightarrow$ origin

$r = 2 \rightarrow$ circle of radius 2 centered at the origin

but only at the points where $x = y$.

5. A particle moves in a velocity field $\mathbf{V}(x, y) = \langle x^2, x + y^2 \rangle$. If it is at position $(2, 1)$ at time $t = 3$, estimate its location at time $t = 3.01$.

[Hint: Use the velocity vector at $t = 3$ to write a linear approximation of the position function.]

$$s(t+dt) \approx s(t) + dt \cdot \vec{V}(s(t))$$

$$= \langle 2,1\rangle + 0.01 \cdot \langle 2^2, 2+1^2\rangle$$

$$= \langle 2,1\rangle + \langle 0.04, 0.03\rangle$$

$$= \langle 2.04, 1.03\rangle$$

6. The **flow lines** (or **stream lines**) of a vector field $\mathbf{F}$ are the paths followed by a particle whose velocity field is the given vector field: $\gamma(t) = \langle x(t), y(t)\rangle$ such that $\dot{\gamma}(t) = \mathbf{F}(x(t), y(t))$. Thus a vector field is tangent to its flow lines.

Find the flow line of the vector field $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$ passing through the point $(1, 1)$.

$$\dot{\gamma}(t) = \langle \dot{x}(t), \dot{y}(t)\rangle \qquad \vec{F}(x,y) = \langle x(t), -y(t)\rangle$$

$$\frac{dx}{dt} = x \qquad \frac{dy}{dt} = -y$$

$$\int \frac{dx}{x} = \int dt \qquad \int \frac{dy}{y} = \int -dt$$

$$e^{\ln|x|} = e^{t+c} \qquad e^{\ln|y|} = e^{-t+c}$$

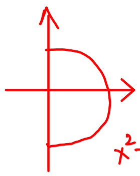
$$x(t) = Ce^{t} \qquad y(t) = Ce^{-t}$$

$$x(0) = 1 \Rightarrow C = 1 \qquad y(0) = 1 \Rightarrow C = 1$$

so $x(t) = e^{t}$ $\qquad y(t) = e^{-t}$

so the flow line is $\gamma(t) = \langle e^{t}, e^{-t}\rangle$

7. Evaluate the path integral $\int_C xy^4\,ds$ where C is the right half of the circle $x^2+y^2=16$.



$x^2+y^2=16$

$x = 4\cos\theta$
$y = 4\sin\theta$
$\theta: -\frac{\pi}{2} \to \frac{\pi}{2}$
$ds = \sqrt{4^2(-\sin\theta)^2 + 4^2\cos^2\theta}\;d\theta$
$= 4\,d\theta$

$u = \sin\theta$ $\quad u(\pi/2) = 1$
$du = \cos\theta\,d\theta$ $\quad u(-\pi/2) = -1$

so, $\int_C xy^4\,ds = 4\int_{-\pi/2}^{\pi/2} 4\cos\theta\, 4^4 \sin^4\theta\, d\theta$

$= 4^6 \int_{-1}^{1} u^4\,du = 4^6 \cdot 2\int_0^1 u^4\,du$

$= 2^{13}\cdot \frac{1}{5}u^5 \Big|_0^1$

$= \frac{2^{13}}{5} = \boxed{\frac{8192}{5}}$

8. Evaluate $\int_C 2x + 9z\,ds$ where C is the path parametrized by $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$, $0 \le t \le 1$.

$ds = \|\dot{\vec{r}}\|\,dt$ $\qquad \dot{\vec{r}} = \langle 1, 2t, 3t^2\rangle$ $\qquad \|\dot{\vec{r}}\| = \sqrt{1+4t^2+9t^4}$

$2x + 9z = 2t + 9t^3$

$u = 1+4t^2+9t^4$ $\quad u(1) = 14$
$du = (8t+36t^3)\,dt$ $\quad u(0) = 1$
$= 4(2t+9t^3)\,dt$

so, $\int_C 2x+9z\,ds = \int_0^1 (2t+9t^3)\sqrt{1+4t^2+9t^4}\,dt$

$= \frac{1}{4}\int_1^{14}\sqrt{u}\,du = \frac{1}{4}\int_1^{14} u^{1/2}\,du$

$= \frac{1}{\cancel{4}\,2}\cdot\frac{\cancel{2}}{3}u^{3/2}\Big|_1^{14} = \boxed{\frac{1}{6}\left(14^{3/2}-1\right)}$

9. Evaluate the path integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = \langle z, y, -x \rangle$ and $\mathbf{r}(t) = \langle t, \sin t, \cos t \rangle$, $0 \le t \le \pi$.

$d\vec{r} = \langle dt, \cos t\, dt, -\sin t\, dt \rangle \qquad \vec{F}(\vec{r}(t)) = \langle \cos t, \sin t, -t \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \underbrace{\int_0^\pi \cos t\, dt}_{I_1} + \underbrace{\int_0^\pi \cos t \sin t\, dt}_{I_2} + \underbrace{\int_0^\pi t \sin t\, dt}_{I_3} = \boxed{\pi}$$

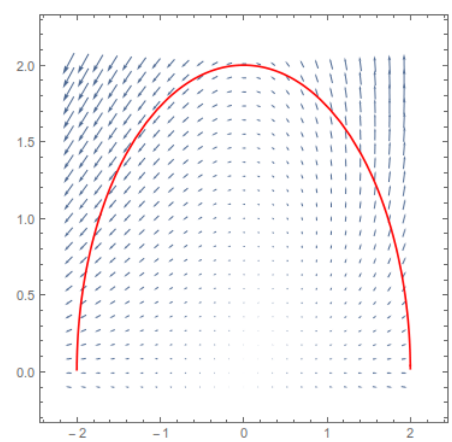
$I_1 = \int_0^\pi \cos t\, dt = \sin t \Big|_0^\pi = \sin\pi - \sin 0 = 0 - 0 = 0$

$I_2 = \int_0^\pi \cos t \sin t\, dt = \left\{ \begin{matrix} u = \sin t & u(\pi) = 0 \\ du = \cos t & u(0) = 0 \end{matrix} \right\} = \int_0^0 u\, du = 0$

$I_3 = \int_0^\pi t \sin t\, dt = -t\cos t + \sin t \Big|_0^\pi = -\pi(-1) - 0 + 0 - 0 = \pi$

10. Use a `CAS` to plot the vector field $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + xy\,\mathbf{j}$ and the curve $C : x^2 + y^2 = 4$ traversed clockwise from $(2, 0)$ to $(-2, 0)$ on the same set of axes.

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.



$\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle \qquad d\vec{r} = \langle -2\sin t\, dt, 2\cos t\, dt \rangle$

$t: 0 \to \pi \qquad \vec{F}(\vec{r}(t)) = \langle 2(\cos t - \sin t), 4\cos t \sin t \rangle$

$\vec{F} \cdot d\vec{r} = (-4 \sin t \cos t + 4\sin^2 t + 8\cos^2 t \sin t)\, dt$

So, $\int_C \vec{F} \cdot d\vec{r} = \underbrace{-4\int_0^\pi \sin t \cos t\, dt}_{I_1} + \underbrace{4\int_0^\pi \sin^2 t\, dt}_{I_2} + \underbrace{8\int_0^\pi \cos^2 t \sin t\, dt}_{I_3}$

$I_1: \left\{ \begin{matrix} u = \sin t & u(\pi) = 0 \\ du = \cos t\, dt & u(0) = 0 \end{matrix} \right\} \int = 0.$

$I_2: \frac{4}{2}\int_0^\pi 1 - \cos 2t\, dt = 2\left(t - \frac{1}{2}\sin 2t\right)\Big|_0^\pi = 2\pi$

$I_3: \left\{ \begin{matrix} u = \cos t & u(\pi) = -1 \\ du = -\sin t\, dt & u(0) = 1 \end{matrix} \right\} = 8\int_{-1}^1 u^2\, du = 16\int_0^1 u^2\, du = \frac{16}{3} u^3 \Big|_0^1 = \frac{16}{3}$

So, $\int_C \vec{F} \cdot d\vec{r} = \boxed{2\pi + \frac{16}{3}}$