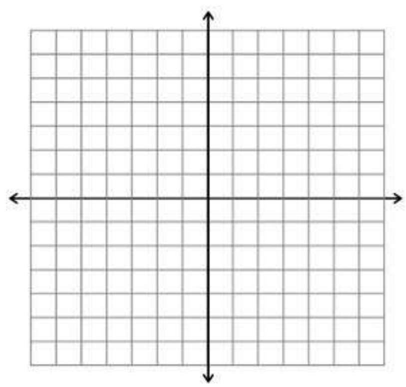
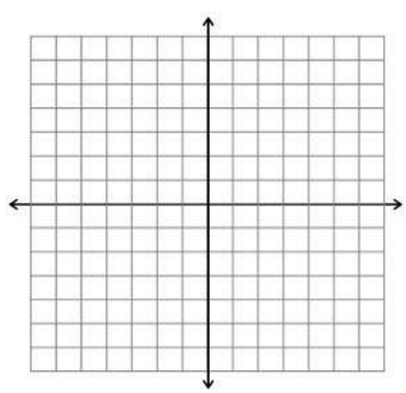
10. Vector Fields and Path Integrals

This week's Good Problems cover material from sections 15.1 and 15.3 of our book. Topics include vector fields and path integrals (known as "line" integrals in the book and in other areas of math, but this name is deceiving).

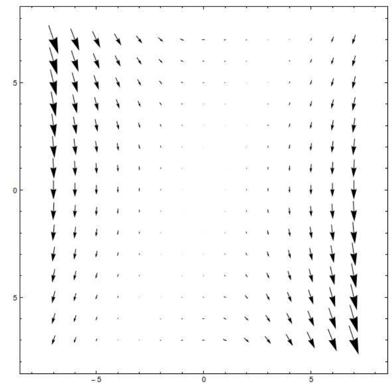
1. Sketch (some of) the vector field $\mathbf{F}(x, y) = \frac{1}{2}(\mathbf{i} + \mathbf{j})$.



2. Sketch (some of) the vector field $\mathbf{F}(x, y) = \left\langle \frac{y}{\sqrt{x^2 + y^2}}, \frac{-x}{\sqrt{x^2 + y^2}} \right\rangle$.



3. Consider the vector field $\mathbf{F}(x,y) = (y^2 - 2xy)\mathbf{i} + (3xy - 6x^2)\mathbf{j}$.



Describe the appearance by finding the set of points (x,y) satisfying $\mathbf{F}(x,y) = \mathbf{0}$.

4. Let $\mathbf{x} = \langle x, y\rangle$, $r = \|\mathbf{x}\|$, and $\mathbf{F}(\mathbf{x}) = (r^2 - 2r)\mathbf{x}$. Use a `CAS` (such as Wolfram|Alpha) to plot the vector field on various domains, then describe its appearance by finding the points where $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.

5. A particle moves in a velocity field $\mathbf{V}(x, y) = \langle x^2, x + y^2 \rangle$. If it is at position $(2, 1)$ at time $t = 3$, estimate its location at time $t = 3.01$.

[Hint: Use the velocity vector at $t = 3$ to write a linear approximation of the position function.]

6. The **flow lines** (or **stream lines**) of a vector field $\mathbf{F}$ are the paths followed by a particle whose velocity field is the given vector field: $\gamma(t) = \langle x(t), y(t) \rangle$ such that $\dot{\gamma}(t) = \mathbf{F}(x(t), y(t))$. Thus a vector field is tangent to its flow lines.

Find the flow line of the vector field $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$ passing through the point $(1, 1)$.

7. Evaluate the path integral $\int_C xy^4\, ds$ where C is the right half of the circle $x^2 + y^2 = 16$.

8. Evaluate $\int_C 2x + 9z\, ds$ where C is the path parametrized by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \leq t \leq 1$.

9. Evaluate the path integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = \langle z, y, -x \rangle$ and $\mathbf{r}(t) = \langle t, \sin t, \cos t \rangle$, $0 \leq t \leq \pi$.

10. Use a `CAS` to plot the vector field $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + xy\,\mathbf{j}$ and the curve $C : x^2 + y^2 = 4$ traversed clockwise from $(2, 0)$ to $(-2, 0)$ on the same set of axes.

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.