Chapter 13: Vector Functions

13.1 Vector Functions and Space Curves

A *vector function* is a function that takes in a real number as ~~its~~ argument, and returns a vector. (Usually in $\mathbb{R}^3$, but possibly in any $\mathbb{R}^n$.)

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$= f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}.$$

The functions f, g, and h are real-valued functions called the *component functions* of $\vec{r}$.

Ex. $\vec{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$

$$\left.\begin{aligned} x &= t^3 \\ y &= \ln(3-t) \\ z &= \sqrt{t} \end{aligned}\right\}$$ these are the component functions for $\vec{r}$.

The domain of $\vec{r}$ is the intersection of the domains of x, y, and z.

dom(x): $\mathbb{R}$
dom(y): $3-t>0 \Rightarrow t<3$
dom(z): $t \geq 0$

Thus the domain of $\vec{r}$ is $0 \leq t < 3$.

Defn. If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

Provided the limits of all component functions exist.

Ex. Find $\lim_{t\to 0} \vec{r}(t)$ where $\vec{r}(t) = \langle 1+t^3, te^{-t}, \frac{\sin t}{t} \rangle$

$$\lim_{t\to 0} \vec{r}(t) = \langle 1, 0, 1 \rangle$$

Def'n. A vector function $\vec{r}(t)$ is continuous at $t=a$ if and only if

$$\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$$

Exc. Show that a vector function is continuous at $t=a$ if and only if each of its component functions is.

There is a close connection between vector functions and space curves.

Suppose $f, g,$ and h ~~vary~~ are continuous on an interval I. Then the set C of all points (x,y,z) in space, where

$$x = f(t), \quad y = g(t), \quad z = h(t), \qquad t \in I \qquad (*)$$

is called a space curve.

Equations $(*)$ are called the parametric equations for C and t is called a parameter.

Think of C as being traced out by a particle whose position at time t is given by the vector

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle.$$

Ex. Describe the curve traced by the vector function.

$$\vec{r}(t) = \langle 1+t, 2+5t, -1+6t \rangle$$

Ex. Sketch the curve whose vector equation is

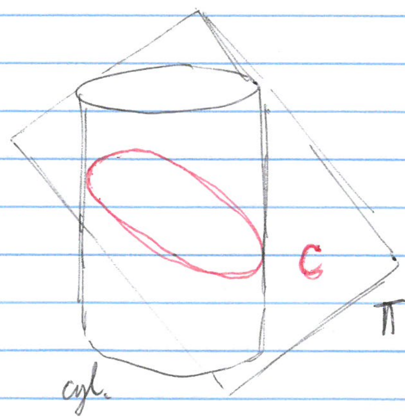
$$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$$

This curve is called a <u>helix</u>.

Ex. Find a vector function that represents the curve of the intersection of the cylinder

$$x^2+y^2=1$$

w/ the plane $y+z=2$



The projection onto the xy-plane is the unit circle, so we can put

$$x=\cos t \qquad y=\sin t \qquad 0\le t\le 2\pi$$

From the eq'n of the plane

$$z = 2-y = 2-\sin t$$

so $\vec{r} = \langle \cos t, \sin t, 2-\sin t\rangle \qquad t\in[0,2\pi)$

This eq'n is called a <u>parametrization</u> of the curve C.

On the computer:

Graph these:

1. Toroidal spiral:

$$\begin{cases} x = (4 + \sin 20t) \cos t \\ y = (4 + \sin 20t) \sin t \\ z = \cos 20t \end{cases}$$

2. Trefoil Knot:

$$\begin{cases} x = (2 + \cos \frac{3}{2}t) \cos t \\ y = (2 + \cos \frac{3}{2}t) \sin t \\ z = \sin \frac{3}{2}t \end{cases}$$

3. Twisted Cubic:

$$\begin{cases} x = t \\ y = t^2 \\ z = t^3 \end{cases}$$