Motion in space: Velocity and Acceleration

Let $\vec{r}$ be a vector function w/ corresponding space curve C. Think of $\vec{r}$ as the position function of a particle in space, and C as its "worldline." The curve C, the worldline, is the curve traced out by the particle over its entire "life".

Defn. The velocity of the particle is the vector function
$$\vec{v}(t) := \lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h} = \dot{\vec{r}}(t)$$
The acceleration of the particle is $\vec{a}(t) = \dot{\vec{v}}(t) = \ddot{\vec{r}}(t)$.
The speed of the ~~position~~ particle is the function $\sigma(t) = \|\dot{\vec{r}}(t)\| = \|\vec{v}(t)\|$

Ex. Let $\vec{r}(t) = \langle t^3, t^2 \rangle$ be the position function of a particle moving in the plane. Find its velocity, acceleration, and speed when $t=1$.

$\dot{\vec{r}}(t) = \langle 3t^2, 2t \rangle$ so $\vec{v}(1) = \langle 3, 2 \rangle$

$\ddot{\vec{r}}(t) = \langle 6t, 2 \rangle$ $\vec{a}(1) = \langle 6, 2 \rangle$

$\sigma(1) = \sqrt{3^2+2^2} = \sqrt{13}$

Ex. Find the velocity, acceleration, and speed of the particle whose motion in space is given by

$$\vec{r}(t) = t^2\vec{i} + e^t\vec{j} + te^t\vec{k}$$

$\vec{v}(t) = \langle 2t, e^t, (1+t)e^t \rangle$

$\sigma(t) = \sqrt{4t^2 + e^{2t} + (1+t)^2 e^{2t}}$

$\vec{a}(t) = \langle 2, e^t, (2+t)e^t \rangle$

Ex. A particle starts at an initial position $\vec{r}(0) = \langle 1, 0, 0 \rangle$ w/ an initial velocity of $\vec{v}(0) = \langle 1, -1, 1 \rangle$. Find the position function if its acceleration is $\vec{a}(t) = 4t\vec{i} + 6t\vec{j} + \vec{k}$.

$\vec{r}(t) = \left(\frac{2}{3}t^3 + t + 1\right)\vec{i} + (t^3 - t)\vec{j} + \left(\frac{1}{2}t^2 + t\right)\vec{k}$

Newton's Second Law of motion

Let $\vec{F}(t)$ be a force field (a vector at each time t) that acts on a moving particle. Newton's 2^{nd} Law says that

$$\vec{F}(t) = m\,\vec{a}(t)$$

where m is the mass of the particle (assumed to be constant for our purposes — not true in relativity.)

Ex. A particle of mass m moves in a circular path w/ constant angular acceleration ω. Its position is given by $\vec{r}(t) = a\cos\omega t\,\vec{i} + a\sin\omega t\,\vec{j}$.

The force acting on the object is then given by $\vec{F}(t) = m\vec{a}(t)$:

$$\vec{a}(t) = \langle -a\omega^2\cos\omega t, -a\omega^2\sin\omega t\rangle$$

so

$$\vec{F}(t) = -am\omega^2\langle\cos\omega t, \sin\omega t\rangle$$

This is $\vec{F}(t) = -m\omega^2\,\vec{r}(t)$. Thus the force due to acceleration points inward toward the origin of the circle.

This is <u>centripetal</u> <u>force</u>.

Tangential and Normal Components of acceleration

Recall $\vec{T} = \frac{\vec{r}'}{\|\vec{r}'\|} = \frac{\vec{v}}{v}$, thus $\vec{v} = v\vec{T}$.

Velocity is always in the direction of the unit tangent.

Acceleration is then given by

$$\vec{a} = \dot{\vec{v}} = (v\vec{T})^\cdot = \dot{v}\vec{T} + v\dot{\vec{T}}$$

From Frenet-Serret, $\kappa = \frac{\|\dot{\vec{T}}\|}{\|\vec{r}'\|} = \frac{\|\dot{\vec{T}}\|}{v} \rightarrow \|\dot{\vec{T}}\| = v\kappa$ and

$\vec{N} = \frac{\dot{\vec{T}}}{\|\dot{\vec{T}}\|} \Rightarrow \dot{\vec{T}} = \kappa v \vec{N}$

Thus, we have a decomposition of acceleration into tangential and normal components:

$$\boxed{\vec{a}(t) = \dot{v}\vec{T} + \kappa v^2 \vec{N}}$$

Frequently, we write $a_T(t) = \dot{v}(t)$ and $a_N(t) = \kappa(t)\, v^2(t)$ so that

$$\vec{a}(t) = a_T \vec{T} + a_N \vec{N}$$

This means that the acceleration of a particle in space always lies in its osculating plane at the point.

Next notice that the acceleration in the tangential direction is the derivative of ~~[illegible]~~ speed. This is the same as Calc I. Moving to 3D introduces normal acceleration $a_N = \kappa v^2$ which is governed by the curvature of the curve.

In relativity, curvature is the force due to gravity. (curvature of spacetime itself, not of a particle moving therein)

Ex. In terms of the position function $\vec{r}$,

$$\boxed{a_T = \frac{\dot{\vec{r}}(t) \cdot \ddot{\vec{r}}(t)}{\|\dot{\vec{r}}(t)\|}}$$

and

$$\boxed{a_N = \frac{\|\dot{\vec{r}} \times \ddot{\vec{r}}(t)\|}{\|\dot{\vec{r}}(t)\|}}$$

Ex. $\vec{r}(t) = \langle t^2, t^2, t^3 \rangle$ Find a_T, a_N and $\vec{a}$.

$$a_T = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}} \qquad a_N = \frac{6\sqrt{2}\,t^2}{\sqrt{8t^2+9t^4}}$$

Kepler's Laws of Planetary Motion

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal area in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of its orbit's major axis.

Recall: Newton's 2^{nd} Law: $\vec{F} = m\vec{a}$

Law of Gravity (Newton): $\vec{F} = -\frac{GMm}{r^3}\vec{r} = -\frac{GMm}{r^2}\vec{u}$

where G is the gravitational constant, M and m are the masses of the objects, r is the distance between their centers, $r = \|\vec{r}\|$, and $\vec{u} = \vec{r}/r$ is a unit vector in the direction of $\vec{r}$.

First we show that the planet moves in a single plane: Setting Newton's equations together, we obtain

$$\vec{a} = \frac{-GM}{r^3}\vec{r}$$

so that $\vec{a}$ is parallel to $\vec{r}$. Thus $\vec{r} \times \vec{a} = \vec{0}$.

Then $(\vec{r}\times\vec{v})^{\cdot} = \dot{\vec{r}}\times\vec{v} + \vec{r}\times\dot{\vec{v}}$

$$= \vec{v}\times\vec{v} + \vec{r}\times\vec{a}$$
$$= \vec{0} + \vec{0}$$
$$= \vec{0}$$

Thus $\vec{r}\times\vec{v} = \vec{h}$ where $\vec{h}$ is a constant vector. We may assume that $\vec{h}\neq\vec{0}$ so that $\vec{r}\perp\vec{h}$ for all values of t. This means that the orbit of the planet lies on a plane through the origin and perpendicular to $\vec{h}$. So the orbit is a plane curve.

To prove Kepler's first law,

$$\vec{h} = \vec{r}\times\vec{v} = \vec{r}\times\dot{\vec{r}} = r\vec{u}\times(r\vec{u})^{\cdot}$$
$$= r\vec{u}\times(\dot{r}\vec{u} + r\dot{\vec{u}}) = r\dot{r}(\vec{u}\times\vec{u}) + r^2(\vec{u}\times\dot{\vec{u}})$$
$$= r^2(\vec{u}\times\dot{\vec{u}})$$

Then $\vec{a}\times\vec{h} = -\frac{GM}{r^2}\vec{u}\times r^2(\vec{u}\times\dot{\vec{u}}) = -GM\left((\vec{u}\cdot\dot{\vec{u}})\vec{u} - (\vec{u}\cdot\vec{u})\dot{\vec{u}}\right)$

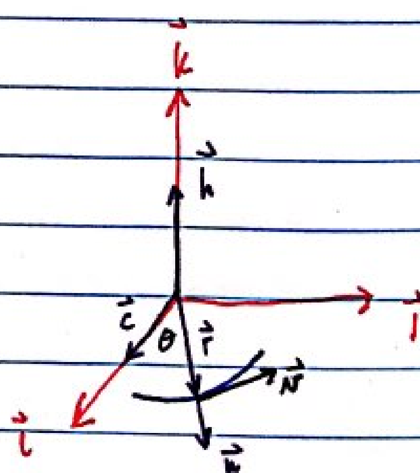
but $\|\vec{u}\|^2 = 1$ and since $\|\vec{u}(t)\| = 1$, $\vec{u}\cdot\dot{\vec{u}} = 0$. Thus

$$\vec{a}\times\vec{h} = GM\dot{\vec{u}}$$

and $$(\vec{v}\times\vec{h})^{\cdot} = GM\dot{\vec{u}}$$

Integrating both sides, $\vec{v}\times\vec{h} = GM\vec{u} + \vec{c}$ where $\vec{c}$ is a constant vector.

Now, choose coordinates so that $\vec{h}$ is in the direction of $\vec{k}$. Now let $\vec{r} = \langle r,\theta\rangle$ be the polar coord's of the planet.



Then $\vec{r}\cdot(\vec{v}\times\vec{h}) = \vec{r}\cdot(GM\vec{u}+\vec{c}) = GM\vec{r}\cdot\vec{u} + \vec{r}\cdot\vec{c}$

$= GMr + r\|\vec{c}\|\cos\theta$

Then $r = \dfrac{\vec{r}\cdot(\vec{v}\times\vec{h})}{GM + c\cos\theta} = \dfrac{1}{GM}\dfrac{\vec{r}\cdot(\vec{v}\times\vec{h})}{1+e\cos\theta}$ where $e = \dfrac{c}{GM}$

But $\vec{r}\cdot(\vec{v}\times\vec{h}) = (\vec{r}\times\vec{v})\cdot\vec{h} = \vec{h}\cdot\vec{h} = \|\vec{h}\|^2 =: h^2$

So, $r = \dfrac{h^2/GM}{1+e\cos\theta} = \dfrac{eh^2/c}{1+e\cos\theta} = \dfrac{ed}{1+e\cos\theta}$

where $d = h^2/c$. This is the polar equation of an ellipse! It has one focus at the origin (the sun) and eccentricity e.

The other two laws can be proved following Stewart's Applied project, p. 896.

Torsion: Show that $\frac{d\vec{B}}{ds} \perp \vec{B}$: Follows from $\|\vec{B}\|=1$.

Show that $\frac{d\vec{B}}{ds} \perp \vec{T}$; $(\vec{T}\times\vec{N})' = \dot{\vec{T}}\times\vec{N} + \vec{T}\times\dot{\vec{N}}$

$$= \vec{0} + \vec{T}\times\dot{\vec{N}}$$

$$\frac{d\vec{B}}{ds} = -\tau(s)\,\vec{N}$$

The function τ is called the <u>torsion</u> of the curve at the point it measures the amount of twisting of the curve.

Torsion is given by $\tau(t) = \dfrac{(\dot{\vec{r}}\times\ddot{\vec{r}})\cdot\dddot{\vec{r}}}{\|\dot{\vec{r}}\times\ddot{\vec{r}}\|^2}$

<u>Ex.</u> Show that the helix $\vec{r}(t) = \langle a\cos t, a\sin t, bt\rangle$ has constant curvature and torsion.