14.3 Partial Derivatives

Defn. Let f be a function of two variables and consider a point (a,b) in the domain of f.

The function $g(x) = f(x,b)$ is a function of only a single variable x. If it is differentiable at a we write

$$f_x(a,b) = \frac{\partial f}{\partial x}(a,b) = \partial_x f(a,b) := g'(a)$$

and call $\frac{\partial f}{\partial x}(a,b)$ the partial derivative of f with respect to x at (a,b).

Similarly,

$$f_y(a,b) = \frac{\partial f}{\partial y}(a,b) = \partial_y f(a,b) := h'(b)$$

where $h(y) = f(a,y)$ is a function of only y. This is the partial derivative wrt y at (a,b).

We can write limit definitions:

$$\begin{cases} \partial_x f(a,b) = \lim\limits_{h\to 0} \dfrac{f(a+h,b) - f(a,b)}{h} \\ \\ \partial_y f(a,b) = \lim\limits_{h\to 0} \dfrac{f(a,b+h) - f(a,b)}{h} \end{cases}$$

OR

$$\begin{cases} \partial_x f(x,y) = \lim\limits_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h} \\ \\ \partial_y f(x,y) = \lim\limits_{h\to 0} \dfrac{f(x,y+h) - f(x,y)}{h} \end{cases}$$

- Rule for finding partial derivatives:

  To find $\partial_x f$, regard y as a constant and differentiate wrt x.
  Similarly, to find $\partial_y f$, regard x as a constant...

Ex. $f(x,y) = x^3 + x^2 y^3 - 2y^2$

Find $\partial_x f$, $\partial_y f$, in general and at $(2,1)$.

Find first partials

Ex. $f(x,y) = x^y$

Ex. $f(x,y) = \frac{x}{y}$

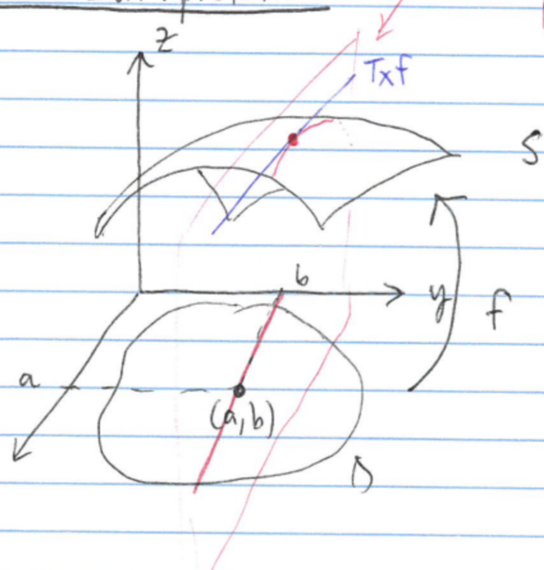
Ex. $f(x,t) = e^{-t}\cos(\pi x)$

Ex. $z = \tan(xy)$

Ex. $g(u,v) = (u^2 v - v^3)^5$

Ex. $w = \frac{e^v}{u+v^2}$

Geometric Interpretation



Slice $y=b$. The intersection of the plane and the surface is a curve. The partial derivative $\partial_x f(a,b)$ is the slope of the tangent line to this curve at $(a,b,f(a,b))$.

Similarly for $\partial_y f(a,b)$, write this out yourself.

Ex. $f(x,y) = 4 - x^2 - 2y^2$

Find $\partial_x f(1,1)$ and $\partial_y f(1,1)$ and interpret these numbers as slopes.

i.e., is the surface "increasing" or "decreasing" in each direction?

Ex. $f(x,y) = \sin\left(\frac{x}{1+y}\right)$

Ex. $x^3 + y^3 + z^3 + 6xyz = 1$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ implicitly.

This extends to functions of multiple variables.

Ex. $f(x,y,z) = e^{xy}\ln(z)$ Find all partials.

Second partial derivatives:

Ex. $f(x,y) = x^3 + x^2y^3 - 2y^2$

find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y \partial x}$.

Thm. Clairaut's Theorem

Suppose f is defined on a disk that contains the point (a,b). If the functions $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are both continuous on D,

then $$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(a,b)} = \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(a,b)}.$$

The partial derivatives commute!

or $f_{yx}(a,b) = f_{xy}(a,b)$.

Finally, Partial Differential Equations.

Laplace's Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Functions u that satisfy this eq'n are called harmonic.

Ex. $u = e^x \sin y$ is a solution.

The wave eq'n:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Ex. $u = \sin(x - at)$ is a sol'n.