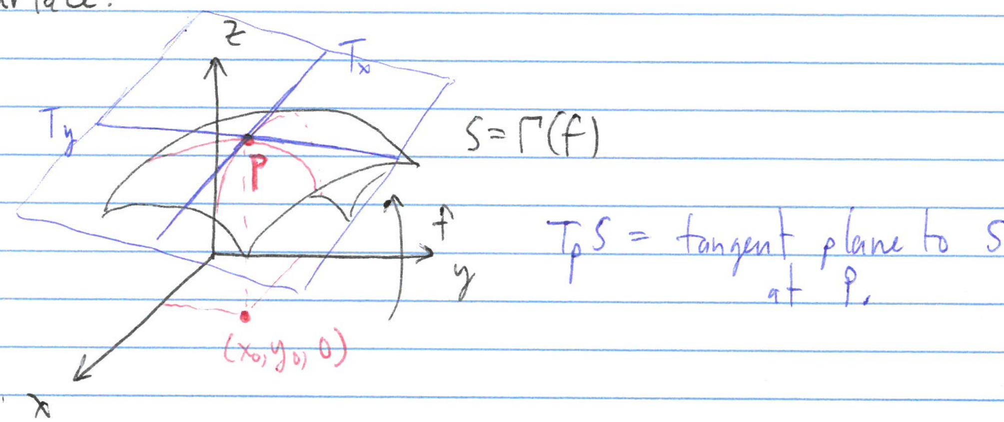
14.4 Tangent Planes and Linear Approximations

Suppose a surface has equation $z=f(x,y)$, where f has continuous partial derivatives, and let $P(x_0,y_0,z_0)$ be a point on the surface.



The partial derivatives $\partial_x f$ and $\partial_y f$ can be used to determine two orthogonal lines that are tangent to the surface $S=\Gamma(f)$ at P.

These lines span a plane through P called the <u>tangent plane</u> to S at P, denoted T_pS.

We want to find an equation of this tangent plane.

If $\vec{n}=\langle A,B,C\rangle$ is normal to the plane, then the equation of T_pS is

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

Dividing through by C and putting $a=-\frac{A}{C}$ and $b=\frac{-B}{C}$, we obtain

$$z=z_0+a(x-x_0)+b(y-y_0)$$

Taking the trace $y = y_0$, we see that

$$z = z_0 + a(x - x_0)$$

is the equation of the tangent line determined by the partial derivative $\partial_x f(x_0, y_0)$, so $a = \partial_x f(x_0, y_0)$.

Similarly, $b = \partial_y f(x_0, y_0)$.

Thus, the plane T_pS has equation

$$T_pS: \quad z = z_0 + \partial_x f(x_0,y_0)(x-x_0) + \partial_y f(x_0,y_0)(y-y_0)$$

Writing it this way, we can regard z as a function of x and y. We write

$$z = T_p f(x,y) = z_0 + \partial_x f(x_0,y_0)(x-x_0) + \partial_y f(x_0,y_0)(y-y_0)$$

$L_p f$

The function $T_p f$ is called the linearization, or linear approximation, of f at p.

If we let $p = (x_0, y_0)$, then

$$\boxed{T_p f(x,y) = f(p) + \partial_x f(p)(x-x_0) + \partial_y f(p)(y-y_0)}$$

* This will also turn out to be the first-order Taylor approximation of f at p.

~~T is called the "tangent functor."~~
~~It can be applied to both maps and their graphs (surfaces). Applied to a surface, it gives the tangent plane to that surface. Applied to a function, it gives the linearization of the function.~~

~~So the tangent functor "linearizes" anything you put into it.~~

Ex. Consider the elliptic paraboloid: $z = 2x^2 + y^2$

Find the linearization $T_p z$ at $P(1,1,3)$, and the eq'n of the tangent plane $T_p S$.

$$\partial_x z = 4x \qquad \partial_x z(p) = 4$$
$$\partial_y z = 2y \qquad \partial_y z(p) = 2$$

so
$$T_p z = 3 + 4(x-1) + 2(y-1)$$
$$= 4x + 2y + 3 - 4 - 2$$

so $\boxed{T_p z = 4x + 2y - 3}$ and $\boxed{T_p S: \quad z - 4x - 2y + 3 = 0}$

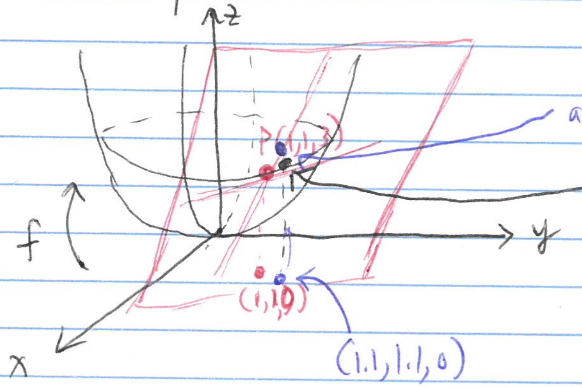
* These are literally two different ways of looking at the same thing.

When (x,y) is "near" p, then $f(x,y) \approx T_p f(x,y)$.

Ex. Estimate the value of $f(1.1, 1.1)$ from the last example.

$$f(1.1, 1.1) \approx T_p f(1.1, 1.1) = 4(1.1) + 2(1.1) - 3$$
$$= 4.4 + 2.2 - 3$$
$$= 6.6 - 3$$
$$= 3.6$$

How do we interpret this?



actual pt on the surface $\Gamma(f)$.

pt on the tangent plane $T_p S$ above $(1.1, 1.1, 0)$. The height is an approximation of the height of $\Gamma(f)$ above $(1.1, 1.1, 0)$.

Ex. Find the linearization of $f(x,y) = 3 + 4x - 9y$ at the point $p(2,1)$

$$f(p) = 3 + 4(2) - 9(1) = 3 + 8 - 9 = 2$$

$$\partial_x f(p) = 4$$

$$\partial_y f(p) = -9$$

so
$$\begin{aligned} L_p f(x) &= f(p) + \partial_x f(p)(x - x_0) + \partial_y f(p)(y - y_0) \\ &= 2 + 4(x-2) + (-9)(y-1) \\ &= 2 + 4x - 8 - 9y + 9 \\ &= 3 + 4x - 9y \\ &= f(x,y) \ ! \end{aligned}$$

→ The linearization of a linear function is itself!

Defn. Let $z = f(x,y)$; then f is <u>differentiable</u> at (a,b) if Δz can be expressed in the form

$$\Delta z = \partial_x f(a,b)\Delta x + \partial_y f(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

⊗ i.e., f is diff'able at $p(a,b)$ if the tangent plane $T_p f$ is a good approximation to f at p.

Theorem. If the partial derivatives f_x and f_y exist near (a,b) and are continuous, then f is diff'able at (a,b).

Ex. $f(x,y) = xe^{xy}$

Show that f is diff'able at $(1,0)$ and find the linearization Lf there. Use it to approximate $f(1.1,-0.1)$

$$\left.\begin{aligned} \partial_x f &= xy\,e^{xy} + e^{xy} = (xy+1)e^{xy} \\ \partial_y f &= x^2 e^{xy} \end{aligned}\right\} \text{both are continuous}$$

$$\left.\begin{aligned} f(1,0) &= 1 \\ \partial_x f(1,0) &= (0+1)e^0 = 1 \\ \partial_y f(1,0) &= 1^2 e^0 = 1 \end{aligned}\right\} \begin{aligned} Lf(x,y) &= 1 + 1(x-1) + 1(y-0) \\ &= x + y \end{aligned}$$

Then $f(1.1,-0.1) \approx Lf(1.1,-0.1) = 1.1 - 0.1 = 1$.

The actual value is $1.1\,e^{-0.11} \approx 0.98542$

Differentials

Calc I versus Calc III.

$dx = \Delta x$
$dy = \Delta y$

$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$

$$\begin{aligned} dz &= Lf(x,y) - f(x,y) \\ &= \cancel{f(x,y)} + \partial_x f(x,y)\cdot\Delta x + \partial_y f(x,y)\,\Delta y - \cancel{f(x,y)} \\ &= \partial_x f(x,y)\,dx + \partial_y f(x,y)\,dy \end{aligned}$$

$$\boxed{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$$

Recall in Calc I:

$$f(x+dx) \approx f(x) + dy$$

Now in Calc III:

$$f(x+dx, y+dy) \approx f(x,y) + dz$$

We can use this to approximate changes in f wrt small changes in the domain:

Ex. Let $z = x^2 + 3xy - y^2$

Find dz.

$$\left.\begin{aligned} \partial_x f &= 2x + 3y \\ \partial_y f &= 3x - 2y \end{aligned}\right\} \quad dz = (2x+3y)\,dx + (3x-2y)\,dy$$

If (x,y) changes from $(2,3)$ to $(2.05, 2.96)$ compare the values of Δz and dz.

$$\begin{aligned} f(2,3) &= 2^2 + 2(2)(3) - 3^2 \\ &= 4 + 12 - 9 \\ &= 7 \end{aligned}$$

$$\left.\begin{aligned} dx &= 0.05 \\ dy &= -0.04 \end{aligned}\right\} \quad \begin{aligned} \partial_x f(2,3) &= 2(2) + 3(3) = 4 + 9 = 13 \\ \partial_y f(2,3) &= 3(2) - 2(3) = 0 \end{aligned}$$

so $\quad dz = 13(0.05) + (0)(-0.04) = 0.65$

and
$$\begin{aligned} f(2.05, 2.96) = f(2+0.05, 3+(-0.04)) &\approx f(2,3) + dz \\ &= 7 + 0.65 \\ &= 7.65 \end{aligned}$$

The actual value of Δz is 0.6449 so
$$f(2.05, 2.96) = 7.6449.$$

Functions of three variables: $w = f(x, y, z)$

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz.$$

Ex. $f(x,y) = 1 + x\ln(xy-5)$ at $(2,3)$

$f(x,y) = x^3y^4$ at $(1,1)$

$f(x,y) = e^{-xy}\cos y$ at $(\pi, 0)$