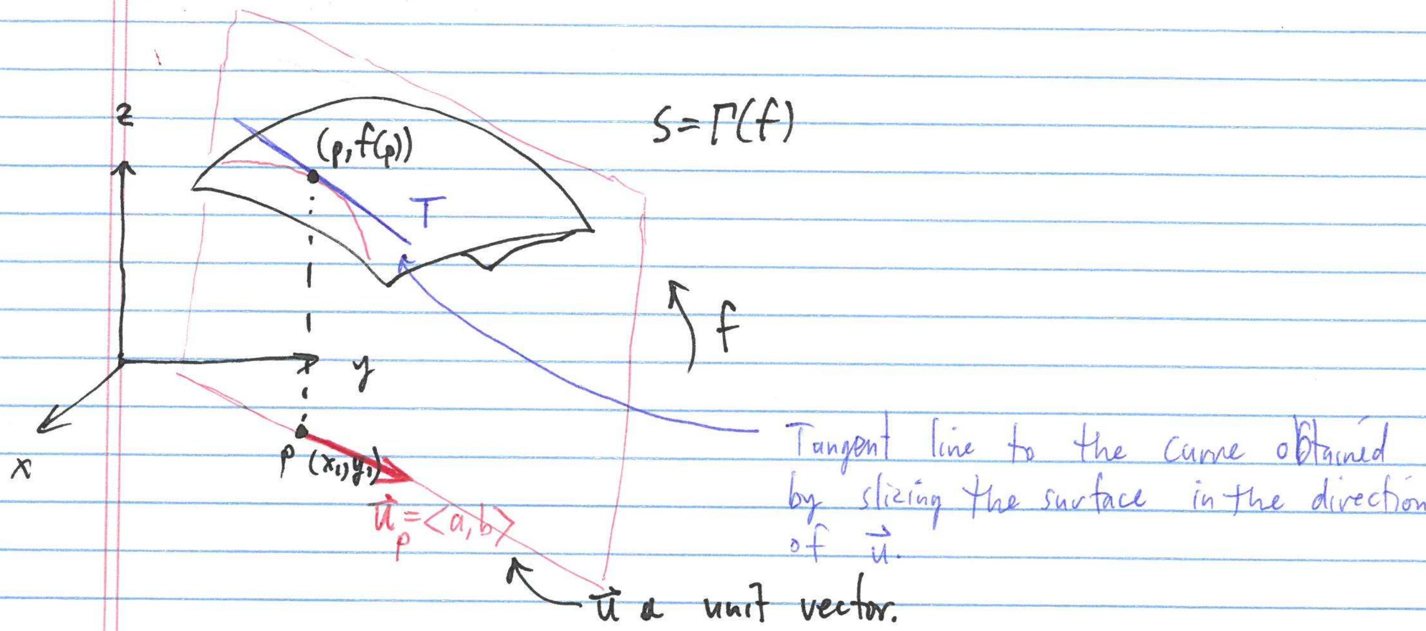
14.6 Directional Derivatives and the Gradient vector



$D_{\vec{u}} f(p)$ = the slope of the tangent line T.

$$D_{\vec{u}} f(p) = \lim_{h \to 0} \frac{f(x_1 + ha, y_1 + hb) - f(x_1, y_1)}{h}$$

provided the limit exists.

__Theorem.__ If f is any smooth function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} = \langle a, b \rangle$ and

$$\boxed{D_{\vec{u}} f(p) = \partial_x f(p) \cdot a + \partial_y f(p) \cdot b}$$

__Proof.__ define a function $g(h) = f(x_1 + ah, y_1 + bh)$.

Then $g'(0) = D_{\vec{u}} f(p)$.

On the other hand $g(h)\,(x,y) = f(x,y)$ where $x = x_0 + ah$, $y = y_0 + bh$

The Chain rule yields $g'(h) = \frac{\partial f}{\partial x} \frac{dx}{dh} + \frac{\partial f}{\partial y} \frac{dy}{dh}$

$$= \partial_x f(\overset{x,y}{p}) \cdot a + \partial_y f(\overset{x,y}{p}) \cdot b$$

Evaluating at $h=0$, $x(0)=x_1$ and $y(0)=y_1$, so

$$D_{\vec{u}}f(p) = g'(0) = \partial_x f(p)\cdot a + \partial_y f(p)\cdot b \quad . \qquad \square$$

If $\vec{u}$ is a unit vector that makes an angle of θ w/ the positive x-axis, then $\vec{u} = \langle \cos\theta, \sin\theta \rangle$ and

$$D_{\vec{u}} f(p) = \partial_x f(p) \cos\theta + \partial_y f(p) \sin\theta .$$

Ex. Find $D_{\vec{u}}f(p)$ if $f(x,y) = x^3 - 3xy + 4y^2$

$\vec{u}$ makes an angle of $\pi/6$ w/ positive x-axis, and $p=(1,2)$.

Get $\dfrac{13-3\sqrt{3}}{2}$.

The Gradient vector

In the formula for directional derivative

$$\begin{aligned} D_{\vec{u}} f(p) &= \partial_x f(p)\cdot a + \partial_y f(p)\cdot b \\ &= \langle \partial_x f(p), \partial_y f(p) \rangle \cdot \langle a, b \rangle \\ &= \langle \partial_x f, \partial_y f \rangle (p) \cdot \vec{u} \end{aligned}$$

The vector $\langle \partial_x f, \partial_y f \rangle = \nabla f$ is called the <u>gradient</u> <u>of f</u>.

we could also write $\operatorname{grad} f = \nabla f$.

Notice ∇f is a function which takes in points in the domain of f, and returns a vector.

In this new notation, we can write

$$\boxed{D_{\vec{u}} f(p) = \nabla f_{(p)} \cdot \vec{u}} \quad \text{or} \quad \boxed{D_{\vec{u}} f = \nabla f \cdot \vec{u}}$$

Ex. $f(x,y) = x^2y^3 - 4y$

Find $\begin{cases} \nabla f \\ \nabla f(2,-1) \\ \text{and } D_{\vec{v}} f(2,-1) \quad \text{where } \vec{v} = 2\vec{i} + 5\vec{j} \end{cases}$

$D_{\vec{v}} f(p) = \frac{32}{\sqrt{29}}$ $\qquad$ $\nabla f(p) = \langle -4, 8 \rangle$

Ex. $f(x,y) = \sin x + e^{xy}$

Find ∇f and $\nabla f(0,1)$.

* The gradient can be extended to functions of more than two variables:

if $F(x,y,z)$, then $\nabla F = \begin{pmatrix} \partial_x F \\ \partial_y F \\ \partial_z F \end{pmatrix} = \langle \partial_x F, \partial_y F, \partial_z F \rangle$

Then we can define $D_{\vec{u}} F = \nabla F \cdot \vec{u}$ for any unit vector $\vec{u}$.