14.5 The Chain Rule

Suppose $z = f(x, y)$, $x = x(t)$, and $y = y(t)$, where x and y are differentiable functions of t, and f is a differentiable function of x and y. Then $z(t) = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Proof on pgs 948-49.

Idea
$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

take limit as $\Delta t \to 0$.

Ex. $z = x^2 y + 3xy^4$ $\quad x = \sin(2t)$ $\quad y = \cos t$

Find $\left.\frac{dz}{dt}\right|_{t=0}$. $= 6.$

Ex. $PV = 8.31T$ gas law

P = pressure, kilo Pascals
V = volume, liters
T = temperature, kelvins

Suppose that when $T = 300K$ and $V = 100L$,

$\frac{dT}{dt} = 0.1\ K/s$ and $\frac{dV}{dt} = 0.2\ L/s$ both increasing.

Find $\frac{dP}{dt}$. Yuck. get ≈ 0.042 kPa/s.

Chain Rule, Part II :

Suppose $z = f(x,y)$ is diff'able of x, y, and $x = x(s,t)$, $y = y(s,t)$ are diff'able of s and t. Then z is diff'able fcn of s and t and:

$$\left\{ \begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &\text{and} \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \end{aligned} \right.$$

Ex. $z = e^x \sin y$, $x = st^2$, $y = s^2 t$

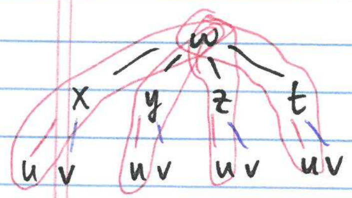
Find $\partial z/\partial s$ and $\partial z/\partial t$

General version of chain rule is analogous.

Ex. $w = f(x,y,z,t)$ $\quad x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, $t = t(u,v)$

Find $\dfrac{\partial w}{\partial u}$.

Make a tree:



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

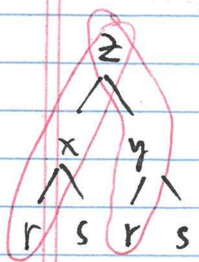
Ex. $u = x^4 y + y^2 z^3$, $x = rse^t$, $y = rs^2 e^{-t}$, $z = r^2 s \sin t$

Find $\frac{\partial u}{\partial s}$ at $(\cancel{x}r, s, t) = (2,1,0)$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} \quad \cdots \qquad \text{get } 192.$$

Ex. Let $z = f(x,y)$, $x = x(r,s)$, $y = y(r,s)$.

Find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial^2 z}{\partial r^2}$



$$\boxed{\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}}$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\right)$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial z}{\partial x}\frac{\partial}{\partial r}\left(\frac{\partial x}{\partial r}\right) + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial}{\partial r}\frac{\partial y}{\partial r}$$

$$= \left(\frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial r} + \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r}\right)\frac{\partial x}{\partial r} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial r^2} +$$

$$+ \left(\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r} + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial r}\right)\frac{\partial y}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial r^2}$$

$$= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial r}\right)^2 + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial r}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial r^2} +$$

$$+ \frac{\partial^2 y}{\partial y^2}\left(\frac{\partial y}{\partial r}\right)^2 + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial r}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial r^2}$$

$$\boxed{\frac{\partial^2 z}{\partial r^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial r}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial r^2} + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial r}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial r^2} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial r}\right)^2}$$

WHOA!

Ex. Use this to calculate $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial^2 z}{\partial r^2}$ for $\left.\begin{array}{l} z = e^x \cos y \\ x = r^2 + s^2 \\ y = rs + \pi \end{array}\right\}$ at ~~$(r,s) = (0,0)$~~ no. $(1,0)$?

$\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial z}{\partial x} = e^x \cos y$ $\qquad \dfrac{\partial x}{\partial r} = 2r$ $\qquad \dfrac{\partial^2 x}{\partial r^2} = 2$

$\dfrac{\partial z}{\partial y} = -e^x \sin y$ $\qquad \dfrac{\partial y}{\partial r} = s$ $\qquad \dfrac{\partial^2 y}{\partial r^2} = 0$

$\dfrac{\partial^2 z}{\partial y^2} = -e^x \cos y$ $\qquad \dfrac{\partial^2 z}{\partial x \partial y} = -e^x \sin y$

Implicit Differentiation

Suppose we have an equation $F(x,y)=0$, and suppose that y is a smooth function of x.

Then $F(x,y) = F(x, y(x)) = F(x) = 0.$ for all $x \in dom(y)$.

by the chain rule, we get

$$\frac{dF}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

but $dx/dx = 1$, so

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

solving for $\frac{dy}{dx}$ we get $\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y} = \frac{-F_x}{F_y}$.

Ex. Find y' for $x^2+y^2=25$

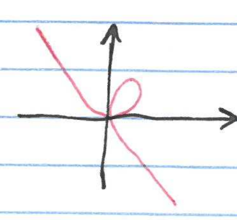
$$F(x,y) = x^2+y^2-25$$

$$\begin{aligned} \partial_x F &= 2x \\ \partial_y F &= 2y \end{aligned} \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

Applying the "old" rule, we get the same answer.

(*) We could have put $F(x,y) = x^2+y^2$ and looked at the level curve $F(x,y)=25$.

Ex. Find y' for $x^3+y^3=6xy$ The Folium of Descartes



Now suppose that $z = f(x,y)$ is given implicitly by the level curve of the function of 3-variables $F(x,y,z) = 0$.

So $F(x,y,f(x,y)) = 0$

Derive a formula for $\frac{\partial z}{\partial x}$ implicitly (involving F's derivatives).

<u>Sol'n</u> by the chain rule,

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$\frac{\partial x}{dx} = 1$ and $\frac{dy}{dx} = 0$, so

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z} = \frac{-F_x}{F_z}.$$

Similarly, $\frac{\partial z}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z} = \frac{-F_y}{F_z}$.

<u>Ex.</u> Find $\partial z/\partial x$ and $\partial z/\partial y$ for $x^3 + y^3 + z^3 + 6xyz = 1$.