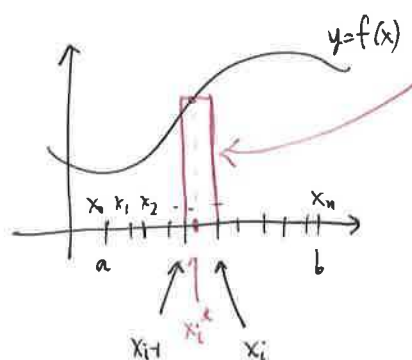


Chapter 12: Multiple Integrals

§ 12.1 Double Integrals over rectangles

As usual, we'll begin by flashing back to Calc I.

The definite integral



$$\begin{aligned}\text{Area of box} &= (x_i - x_{i-1}) \cdot f(x_i^*) \\ &= f(x_i^*) \Delta x_i\end{aligned}$$

where x_i^* is any point in the interval $[x_{i-1}, x_i]$.

Then the area under $y=f(x)$ between a and b is approximately

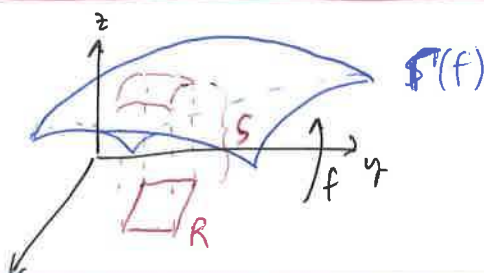
$$\sum_{i=1}^n f(x_i^*) \Delta x_i$$

And exactly

(Riemann Sum) $\longrightarrow \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x_i$ if the limit exists.

This exact area is the definite integral of f from a to b .

Now we want to generalize this to consider volume under a surface. We will spend some time developing the details so that the idea is clear as we move forward. First, we want to find the volume under a surface over a rectangle in the domain.



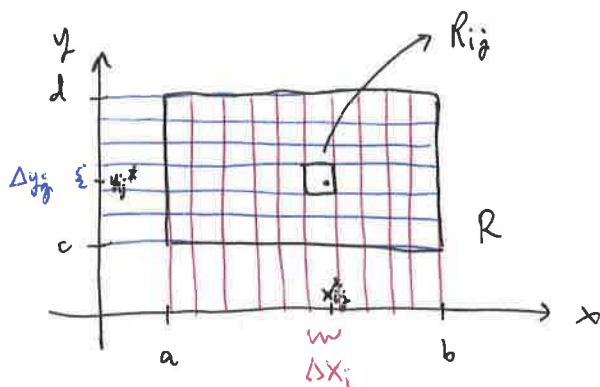
To simplify the situation at first, assume f is a positive function.

$$R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\} = [a, b] \times [c, d]$$

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in R, p \leq z \leq f(x, y)\} \quad \text{solid "over" } R$$

We want to find the volume of S .

Start by partitioning the domain R :



Partition $[a, b]$ into n intervals and $[c, d]$ into m intervals.

This creates an array of $m \times n$ subrectangles of R .

$$\left. \begin{array}{l} \Delta x_i = x_i - x_{i-1} \\ \Delta y_j = y_j - y_{j-1} \end{array} \right\} \Delta A_{ij} = \Delta x_i \Delta y_j \text{ is the area of } R_{ij}.$$

We will estimate the volume ~~of~~ ^{under} the surface above R_{ij} by a box (3D).

To estimate the height of the surface, we need to pick a point $(x_{ij}^*, y_{ij}^*) \in R_{ij}$

and find $f(x_{ij}^*, y_{ij}^*) = z_{ij}^*$.

Then the volume of S_{ij} is approximately $z_{ij}^* \Delta A_{ij} = f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$

Just like in Calc 1, we add up all of the volumes S_{ij} to approximate the volume of S :

$$\begin{aligned} \text{Vol}(S) &\approx \sum_{i=1}^n \sum_{j=1}^m \text{Vol}(S_{ij}) = \sum_{i=1}^n \sum_{j=1}^m f(x_{ij}^*, y_{ij}^*) \Delta A_{ij} \\ &= \sum_{i=1}^n \sum_{j=1}^m f(x_{ij}^*, y_{ij}^*) \Delta y_j \Delta x_i \\ &= \sum_{j=1}^m \sum_{i=1}^n f(x_{ij}^*, y_{ij}^*) \Delta x_i \Delta y_j \end{aligned}$$

As you can guess, we now find the exact volume of S by ~~taking~~ taking the limit as $\Delta x_i, \Delta y_j \rightarrow 0$ (or $n, m \rightarrow \infty$):

$$\text{Vol}(S) = \lim_{\max \Delta x_i, \Delta y_j \rightarrow 0} \sum_i \sum_j z_{ij}^* \Delta A_{ij}$$

Definition. The double integral of f over the rectangle R is

$$\iint_R f(x, y) dA = \lim_{m, n \rightarrow \infty} \sum_{j=1}^m \sum_{i=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$$

← "Double Riemann Sum"

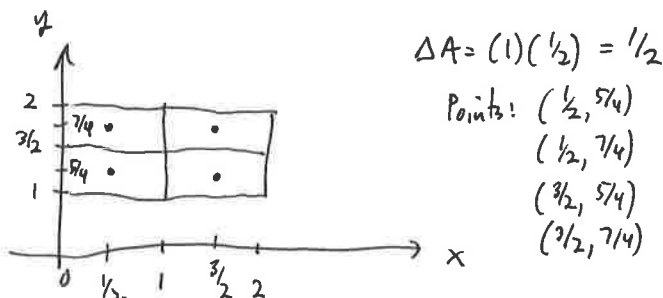
if the limit exists, where ΔA is fixed. (take a regular partition.)

A function is called integrable if this limit exists.

Ex. Use the midpoint rule to ~~evaluate~~ estimate

$$\iint_R (x - 3y^2) dA \quad \text{where } R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$$

w/ $m = n = 2$.



$$\begin{aligned} \iint_R (x - 3y^2) dA &\approx \frac{1}{2} \left[\frac{1}{2} - \frac{75}{16} + \frac{1}{2} - \frac{147}{16} + \frac{3}{2} - \frac{75}{16} + \frac{3}{2} - \frac{147}{16} \right] \\ &= \frac{1}{2} \left[4 - \frac{444}{16} \right] = \left[\frac{-95}{8} \right] = -11.875 \end{aligned}$$

Iterated Integrals:

Recall that integrals are difficult to compute by the definition, but the evaluation thm (FTC part 2) is "very" ~~is~~ simple (well... simpler).

Suppose that ~~defn~~ f is a continuous function of x and y on $R = [a, b] \times [c, d]$. We write $\int_c^d f(x, y) dy$ to mean that f is ~~considered~~ ^{integrated} as a function of y and x is held fixed.

This is called partial integration.

Now

$$A(x) = \int_c^d f(x, y) dy$$

is a function of only x .

We can now integrate A wrt x :

$$\begin{aligned} \int_a^b A(x) dx &= \int_a^b \left[\int_c^d f(x, y) dy \right] dx \\ &= \int_a^b \int_c^d f(x, y) dy dx \end{aligned}$$

We could do the same thing in the reverse order:

$$\int_c^d \int_a^b f(x, y) dx dy.$$

Ex. Evaluate: $\int_0^3 \int_1^2 x^2 y dy dx$ and $\int_1^2 \int_0^3 x^2 y dx dy$

This leads us to:

Fubini's Theorem. If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$$

More generally, this is true if f is bdd on R , f is discontinuous on only a finite number of ^{smooth} curves, and the iterated integrals exist.

Ex. Evaluate $\iint_R (x - 3y^2) dA$ where $R = [0, 2] \times [1, 2]$.

get -12 .

Ex. $\iint_R y \sin(xy) dA$, $R = [1, 2] \times [0, \pi]$

Do $\iint y \sin(xy) dx dy$

The other way is much tougher.

Get 0.

Ex. In the special case when $f(x, y) = g(x)h(y)$ then

$$\iint_R f(x, y) dA = \int_c^d \int_a^b g(x)h(y) dx dy = \int_a^b g(x) dx \int_c^d h(y) dy.$$

Such a function f is called separable.

Ex. Evaluate $\iint_R \sin x \cos y \, dA$ where $R = [0, \pi/2] \times [0, \pi/2]$

$$= \int_0^{\pi/2} \sin x \, dx \int_0^{\pi/2} \cos y \, dy$$

$$= [-\cos \pi/2 + \cos 0] [\sin \pi/2 - \sin 0]$$

$$= [1][1] = 1$$

(25) Ex. $\iint_R xye^{x^2y} \, dA$ $R = [0,1] \times [0,2]$

(26) Ex. $\iint_R \frac{x}{1+xy} \, dA$ $R = [0,1] \times [0,1]$

* Integrating over general regions is a ~~other~~ generalization of this idea.
We need to be careful about the region though.