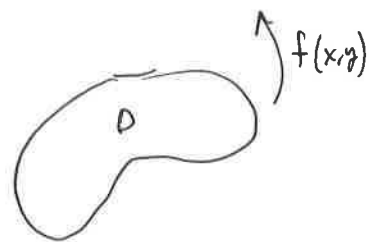
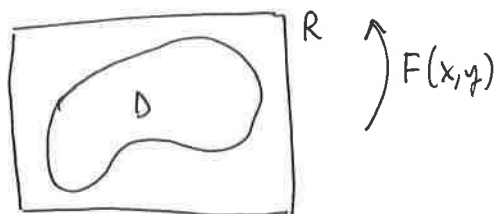


§12.2: Double Integrals over general regions

We want to integrate over regions like:
and not just over rectangles.



The idea is to enclose the region D in a rectangle R:



and define a new function $F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \in R \text{ but } (x, y) \notin D. \end{cases}$

then

$$\boxed{\iint_D f(x, y) \, dA = \iint_R F(x, y) \, dA}$$

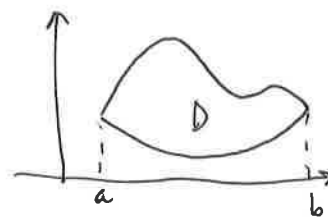
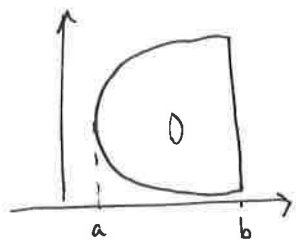
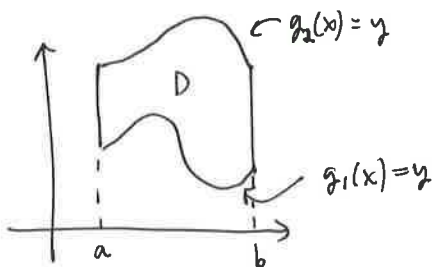
Notice that this idea makes sense no matter how we choose the rectangle R, as long as it contains all of D.

Defn. A plane region D is said to be of Type I iff

$$D = \{ (x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \, g_1(x) \leq y \leq g_2(x) \}$$

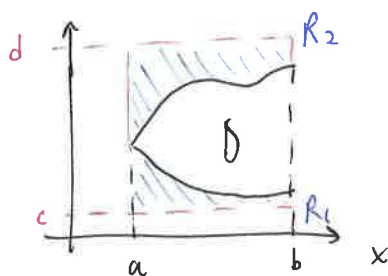
i.e., if it lies between the graphs of two continuous functions $g_1(x)$ and $g_2(x)$ on $[a, b]$.

Examples of Type I regions:



To evaluate a type I integral we choose a rectangle

$$R = [a, b] \times [c, d] \quad \text{with} \quad d \geq g_2(x) \quad \forall x, \quad c \leq g_1(x) \quad \forall x.$$



Then

$$\iint_D f(x, y) dA = \int_a^b \int_c^d F(x, y) dy dx$$

$$= \int_a^b \left[\int_c^d F(x, y) dy \right] dx$$

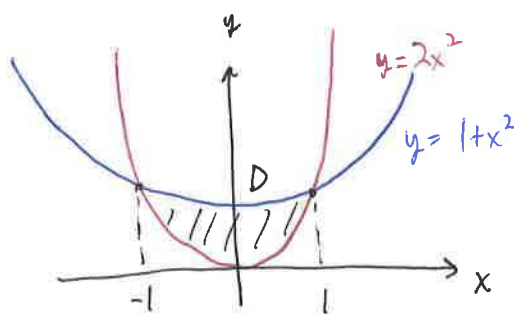
$$(*) \quad = \int_a^b \left[\int_c^{g_1(x)} \cancel{F(x, y)} dy + \int_{g_1(x)}^{g_2(x)} F(x, y) dy + \int_{g_2(x)}^d \cancel{F(x, y)} dy \right] dx$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

For a continuous f on a type I region, we will calculate the integral by using this formula.

Ex. Evaluate $\iint_D (x+2y) dA$ where D is bdd by $y = 2x^2$ and $y = 1+x^2$

First draw the region:



$x \in [-1, 1]$
~~but the region is~~
~~symmetric.~~
 function is odd
 though"

$$\iint_D (x+2y) dA = \iint_D (x+2y) dA$$

$$= \int_{-1}^1 \int_{2x^2}^{1+x^2} (x+2y) dy dx$$

$$= \int_{-1}^1 \left[xy + y^2 \right]_{2x^2}^{1+x^2} dx$$

$$= \int_{-1}^1 \left[x(1+x^2) - 2x^3 + (1+x^2)^2 - 2^2 x^4 \right] dx$$

$$= \int_{-1}^1 \left[x + x^3 - 2x^3 + 1 + 2x^2 + x^4 - 4x^4 \right] dx$$

$$= \int_{-1}^1 \left[1 + x + 2x^2 - x^3 - 3x^4 \right] dx$$

$$= \left[x + \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 - \frac{3}{5}x^5 \right]_{-1}^1$$

$$= \left(1 + \frac{1}{2} + \frac{2}{3} - \frac{1}{4} - \frac{3}{5} \right) - \left(-1 - \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{3}{5} \right) = \frac{60+30+40-15-36}{60} = \frac{130-51}{60}$$

$$= \frac{64}{30} = \boxed{\frac{32}{15}}$$

Type II regions are bounded by continuous functions
 $x = h_1(y)$ and $x = h_2(y)$ on the "left" and "right".

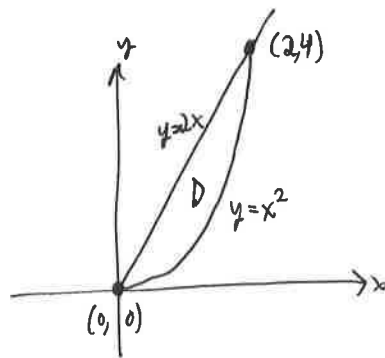
$$D = \{ (x, y) : h_1(y) \leq x \leq h_2(y), c \leq y \leq d \}$$

where h_1, h_2 are defined on $[c, d]$.

In this case we get:

$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

Ex. $\iint_D x^2 + y^2 dA$ w/ D bdd between $y = 2x$ and $y = x^2$.



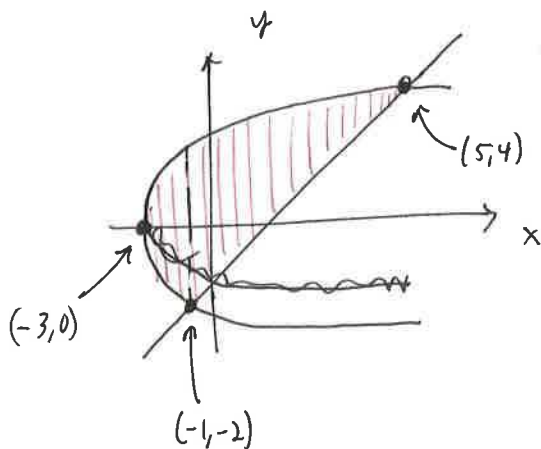
This D can be either a type I or type II region. Let's try it as type II.

"upper" bd: $x = \sqrt{y}$
 lower bd: $x = \frac{1}{2}y$, $y \in [0, 4]$

$$\begin{aligned} \iint_D f(x, y) dA &= \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} x^2 + y^2 dx dy \\ &= \int_0^4 \left[x^2 y + \frac{1}{3} x^3 \right]_{\frac{1}{2}y}^{\sqrt{y}} dy = \int_0^4 y^{3/2} - \frac{1}{2} y^3 + \frac{1}{3} y^{3/2} - \frac{1}{6} y^3 dy \\ &= \int_0^4 \left(\frac{4}{3} y^{3/2} - \frac{2}{3} y^3 \right) dy = \left[\frac{2}{5} \cdot \frac{4}{3} y^{5/2} - \frac{2}{12} y^4 \right]_0^4 = \frac{8}{15} (32) - \frac{4^4}{6} = \boxed{\frac{216}{35}} \end{aligned}$$

Ex. $\iint_D xy \, dA$ where D is bdd by $y=x-1$ and $y^2=2x+6$.

Draw the region:



To do this as a type I region we would have to break it into 2 pieces. Instead, we'll do it as type II.

$$\begin{cases} \text{UB: } x = y + 1 \\ \text{LB: } x = \frac{1}{2}y^2 - 3 \\ \text{dom: } y \in [-2, 4] \end{cases}$$

$$\begin{aligned} \iint_D xy \, dA &= \int_{-2}^4 \int_{\frac{1}{2}y^2 - 3}^{y+1} xy \, dx \, dy \\ &= \int_{-2}^4 \left[\frac{1}{2} x^2 y \right]_{\frac{1}{2}y^2 - 3}^{y+1} dy \\ &= \int_{-2}^4 \left[\frac{1}{2} (y+1)^2 y - \frac{1}{2} \left(\frac{1}{2}y^2 - 3 \right)^2 y \right] dy \\ &= \int_{-2}^4 \left(\frac{1}{2} y^2 + y + \frac{1}{2} \right) y - \left(\frac{1}{8} y^4 + \frac{3}{2} y^2 - \frac{9}{2} \right) y \, dy \\ &= \int_{-2}^4 \frac{1}{2} y^3 + y^2 + \frac{1}{2} y - \frac{1}{8} y^5 + \frac{3}{2} y^3 - \frac{9}{2} y \, dy \\ &= \int_{-2}^4 -4y + y^2 + 2y^3 - \frac{1}{8} y^5 \, dy \\ &= \left[-2y^2 + \frac{1}{3} y^3 + \frac{1}{2} y^4 - \frac{1}{48} y^6 \right]_{-2}^4 = \dots = 36 \end{aligned}$$

Ex. Find the volume of the tetra-hedron bounded by

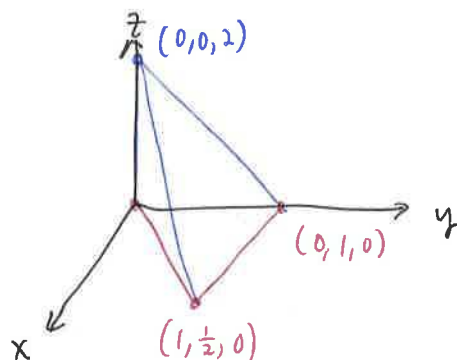
$$x + 2y + z = 2 \Rightarrow z = 2 - x - 2y$$

$$x = 2y$$

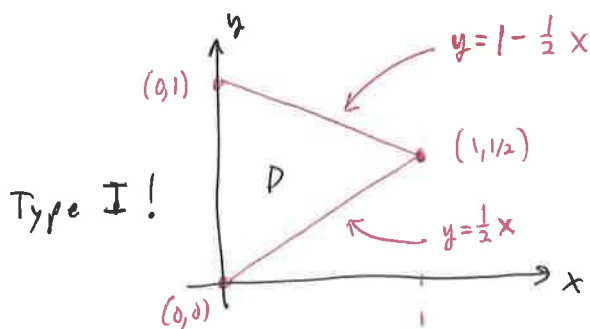
$$x = 0$$

$$z = 0$$

The solid:



The plane region:



$$f(x,y) = 2 - x - 2y$$

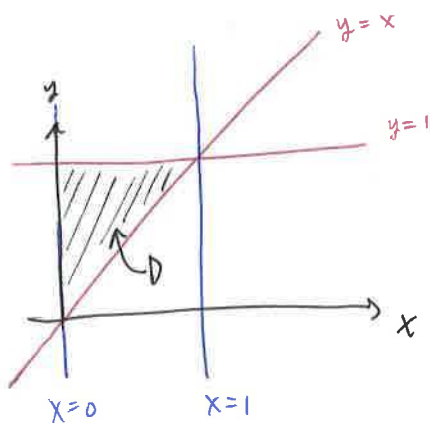
$$V = \iint_D f(x,y) dA = \int_0^1 \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} (2 - x - 2y) dy dx$$

FTIS : Evaluate it. (RE)

Ex. Evaluate: $\int_0^1 \int_x^1 \sin(y^2) dy dx$

Problem: we have no idea what the antiderivative of $\sin(y^2)$ is.

Idea: Think of it as an area integral over a region



This D can be type I or type II.

The integral is written as type I, so we can try to rewrite it as type II.

$$\int_0^1 \int_x^1 \sin(y^2) dy dx = \int_0^1 \int_0^y \sin(y^2) dx dy$$

$$= \int_0^1 \left[x \sin y^2 \right]_0^y dy = \frac{1}{2} \int_0^1 2y \sin y^2 dy$$

$$u = y^2$$

$$du = 2y dy$$

$$= \frac{1}{2} \int_0^1 \sin u du = \frac{1}{2} (-\cos u) \Big|_0^1 = -\frac{1}{2} \cos 1 + \frac{1}{2} \cos 0$$

$$= \boxed{\frac{1}{2} - \frac{1}{2} \cos 1}$$

"

Some properties of Double Integrals:

Usual stuff:

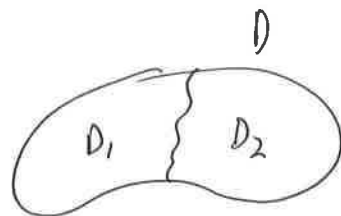
$$\iint_D f \pm g \, dA = \iint_D f \, dA \pm \iint_D g \, dA$$

$$\iint_D c f \, dA = c \iint_D f \, dA$$

If $f \leq g$ for all $(x, y) \in D$ then

$$\iint_D f \, dA \leq \iint_D g \, dA$$

If $D = D_1 \cup D_2$ w/ $D_1 \cap D_2 = \emptyset$,



then

$$\iint_D f \, dA = \iint_{D_1} f \, dA + \iint_{D_2} f \, dA.$$

If $A(D)$ = area of D , then

$$\iint_D 1 \, dA = A(D) \quad (\text{prove it!})$$

If $m \leq f(x, y) \leq M$ for all $(x, y) \in D$ then

$$m A(D) \leq \iint_D f(x, y) \, dA \leq M A(D) \quad (\text{prove it!})$$