§12.5 : Triple Integrals

Just like we can define single integrals for functions of one variable and double integrals for functions of two, we can define triple integrals for functions of 3.

Start with a box

$$B = \{ (x,y,z) \in \mathbb{R}^3 \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s \}.$$

and a function $f(x,y,z) = w$

Using a Riemann sum argument we define

$$\iiint_B f(x,y,z)\, dV = \lim_{\Delta x \Delta y \Delta z \to 0} \sum_i \sum_j \sum_k f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta V_{ijk}$$

Fubini's Theorem. If $f(x,y,z)$ is continuous on the box $B = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint_B f(x,y,z)\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz$$

* the order of the iterated integrals can be changed, as usual.

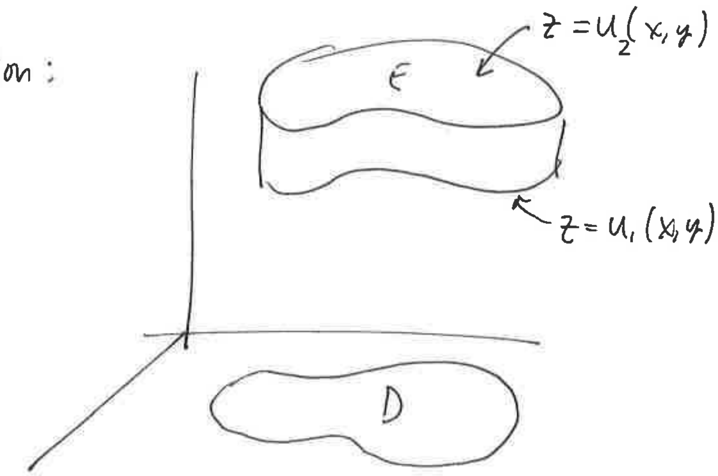
Ex. $\iiint_B xyz^2 \, dV$, $B = [0,1] \times [-1,2] \times [0,3]$

$$= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 \, dx \, dy \, dz$$

$$= \int_0^3 \int_{-1}^2 \frac{1}{2} x^2 y z^2 \Big|_0^1 \, dy \, dz \quad \cdots$$

$$= \frac{27}{4}$$

What about a general region:



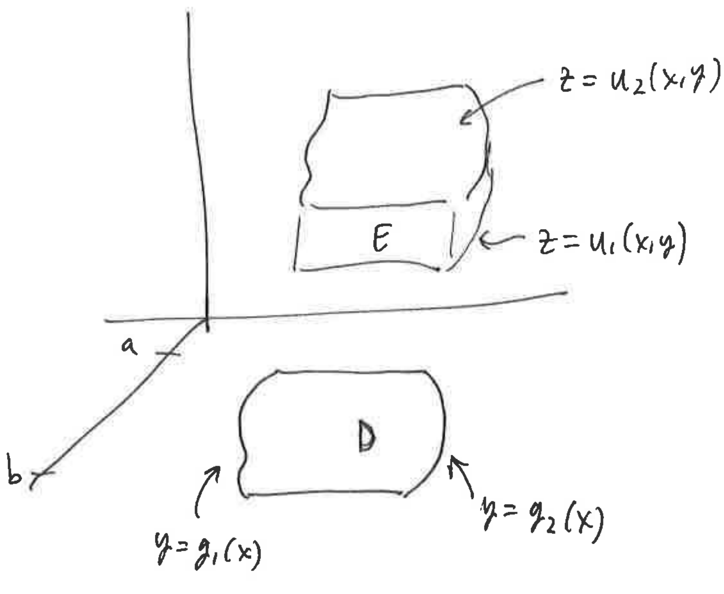
u_1, u_2 continuous on D.

$$E = \{ (x,y,z) \mid (x,y) \in D, \ u_1 \le z \le u_2 \}$$

Then $$\iiint_E f(x,y,z) \, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z) \, dz \right] dA$$

We can reduce the triple integral to a double.

Ex.

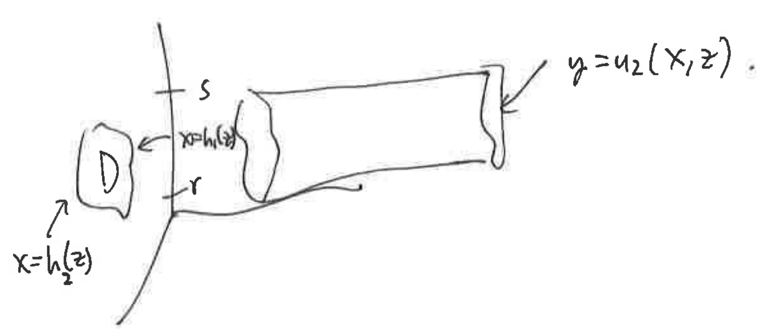


$$\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dA$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

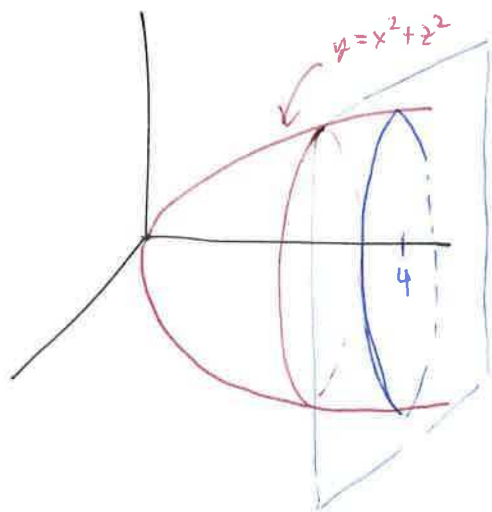
Similarly for a type II D.

A Type III region is one in which z is "rectangular" and $y = u_1(x,z)$, $y = u_2(x,z)$ or $x = u_1(y,z)$, $x = u_2(y,z)$.

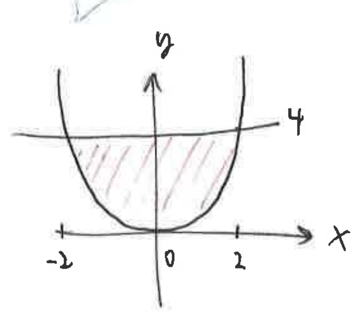


Ex. $\displaystyle\iiint_E \sqrt{x^2+z^2}\, dV$

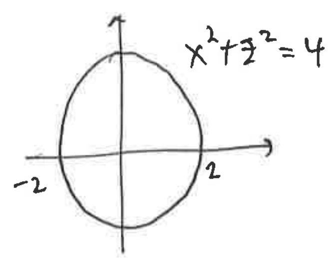
E is bounded by $y = x^2+z^2$ and $y=4$.



In the xy-plane:



In the xz plane $D =$



So $\displaystyle\iiint_E \sqrt{x^2+z^2}\, dV = \iint_D \left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy \right] dA$

$$= \iint_D (4 - x^2 - z^2)\sqrt{x^2+z^2}\, dA$$

Now this D smells like polar coords

$$x = r\cos\theta$$
$$z = r\sin\theta$$
$$r = x^2 + z^2$$

$$\therefore \int_0^{2\pi}\int_0^{2} (4-r^2)\, r \; r \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} 4r^2 - r^4 \; dr\, d\theta$$

$$= \int_0^{2\pi} \frac{4}{3}r^3 - \frac{1}{5}r^5 \Big|_0^2 \; d\theta$$

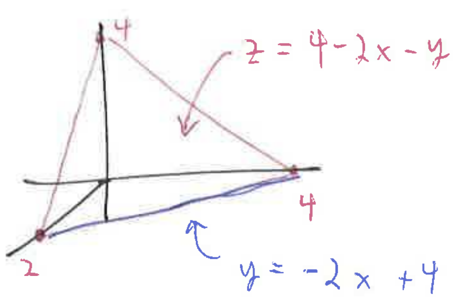
$$= \int_0^{2\pi} \frac{32}{3} - \frac{32}{5} \; d\theta$$

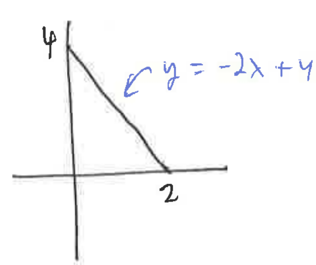
$$= \frac{64}{15}(2\pi) = \boxed{\frac{128}{15}\pi}$$

Sweet! :)

Fact $\quad V(E) = \iiint_E 1 \, dV \quad$ as w/ areas.

<u>Ex</u> (17) Find the volume of the tetrahedron enclosed by the coordinate planes and $2x + y + z = 4$.





$$V(E) = \int_0^2 \int_0^{-2x+4} \int_0^{4-2x-y} 1 \, dz \, dy \, dx$$

Compute.

Ex. (26) Sketch the solid whose volume is given by

$$V(E) = \int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx \, dz \, dy$$

$x = 0$ $\qquad$ $z = 2-y$ $\qquad$ $y = 2$

$x = 4-y^2$ $\qquad$ $z = 0$ $\qquad$ $y = 0$