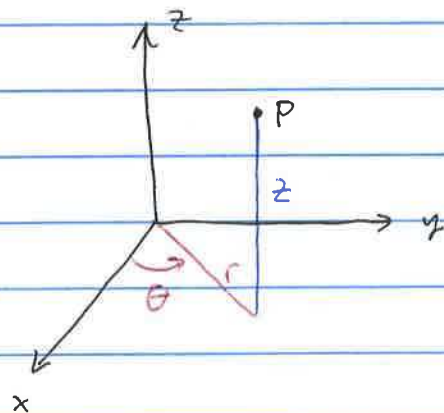


Section 12.6: Triple Integrals in Cylindrical Coordinates

Recall: In our last example of §12.5 we evaluated the z -integral, then found it would be easier to use polar coords to evaluate the xy -integral. We actually used Cylindrical Coordinates:



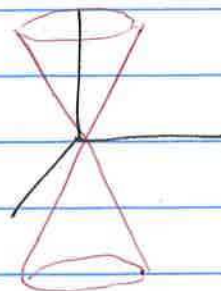
$$\begin{aligned} P &= (r, \theta, z) \\ x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

Ex. Convert $(2, 2\pi/3, 1)$ to rectangular coords.

$$\left. \begin{aligned} x &= 2 \cos\left(\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2}\right) = -1 \\ y &= 2 \sin\left(\frac{2\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3} \\ z &= 1 \end{aligned} \right\} (-1, \sqrt{3}, 1).$$

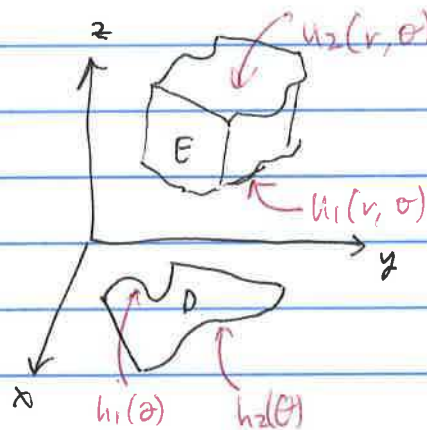
Ex. Describe the surface $z=r$.

It's a cone!



Evaluating triple integrals in cylindrical. A usual argument (make a rectangle enclosing the domain, then "shrink it") yields:

$$\iiint_E f(x, y, z) dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r, \theta)}^{u_2(r, \theta)} f(r, \theta, z) r dz dr d\theta$$



Ex. A solid lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$.

Find the volume of the solid

$$V(E) = \iiint_E 1 dV. \quad \begin{array}{lll} \theta = 0 & r = 0 & z = 1 - r^2 \\ \theta = 2\pi & r = 1 & z = 4 \end{array}$$

$$V(E) = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 1 r dz dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 4r - (r - r^3) dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 3r + r^3 dr d\theta$$

$$= \int_0^{2\pi} \left. \frac{3}{2} r^2 + \frac{1}{4} r^4 \right|_0^1 d\theta$$

$$= \int_0^{2\pi} 7/4 d\theta$$

$$= 7/4 (2\pi) = \boxed{\frac{7}{2} \pi}$$

Ex. Evaluate $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2+y^2) dz dy dx$

$$f(r, \theta, z) = r^2$$

$$z: r \rightarrow 2$$

$$y: -\sqrt{4-x^2} \rightarrow \sqrt{4-x^2}$$

$$\text{i.e., } r: 0 \rightarrow 2$$

$$\theta: 0 \rightarrow 2\pi$$

so the integral becomes:

$$\begin{aligned} & \int_0^{2\pi} \int_0^2 \int_r^2 r^2 r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^2 r^3(2-r) dr d\theta \\ &= \int_0^{2\pi} \int_0^2 2r^3 - r^4 dr d\theta \\ &= \int_0^{2\pi} \left. \frac{1}{2}r^4 - \frac{1}{5}r^5 \right|_0^2 d\theta \\ &= 2\pi \left(8 - \frac{32}{5} \right) = \boxed{16\pi/5} \end{aligned}$$