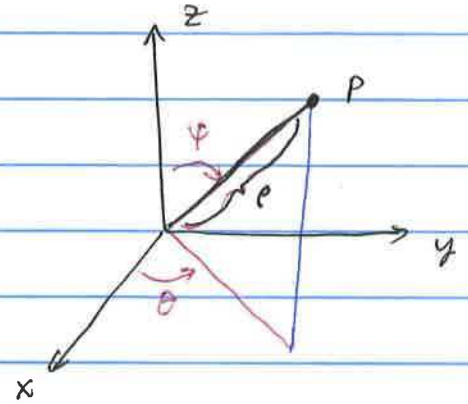
§12.7 : Triple Integrals in Spherical Coords



any point $P \in \mathbb{R}^3$ lies on a sphere centered at the origin of radius $\rho = \sqrt{x^2+y^2+z^2}$.

Cut the sphere with a plane Π through $\vec{0}$ and $\vec{P}$, $\perp$ to xy-plane, this gives the angle θ.

Then φ is the angle in the plane Π that gives P.

To make coordinates unique we demand that

$$\rho \geq 0$$
$$0 \leq \varphi \leq \pi$$
$$0 \leq \theta \leq 2\pi$$

$\rho = c$: sphere w/ radius c.

$\theta = c$: a half-plane (since $\rho \geq 0$)

$0 < \theta < \pi/2$, $\varphi = c$ a half-cone (upper)

$\pi/2 < c < \pi$ (lower half)

Conversions. $x = \rho \sin\varphi \cos\theta$
$y = \rho \sin\varphi \sin\theta$
$z = \rho \cos\varphi$

$$\left.\begin{aligned} z &= \rho \cos\varphi \\ r &= \rho \sin\varphi \end{aligned}\right\} \text{ convert to polar, then to spherical.}$$

or cylindrical

Ex. convert $(0, 2\sqrt{3}, -2)$ to spherical coords.

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{0+12+4} = \sqrt{16} = 4$$

so then $0 = 4 \sin\varphi \cos\theta$

$2\sqrt{3} = 4 \sin\varphi \sin\theta$

$-2 = 4 \cos\varphi \Rightarrow \cos\varphi = -\frac{1}{2} \Rightarrow \varphi = \frac{2\pi}{3}$

then $\cos\theta = 0$, $\sin\varphi \neq 0 \Rightarrow \theta = \frac{\pi}{2}$.

and $\theta \neq \frac{3\pi}{2}$ because $y = 2\sqrt{3} > 0$.

Thus $(0, 2\sqrt{3}, 2)$ is $\left(\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}\right)$ in spherical (ρ, θ, φ) coords.

The volume element in spherical coords:

$$\boxed{dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi} \quad (*) \quad \text{Proj } 4?$$

Then, if E is a spherical wedge (or spherical rectangle) then

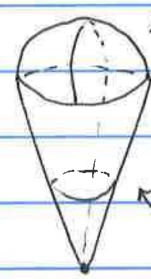
$$\boxed{\iiint_E f(x,y,z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho,\theta,\varphi)\, \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi.}$$

Ex. Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}} \, dV$ w/ $B = \{(x,y,z) \mid x^2+y^2+z^2 \leq 1\}$.

$f(\rho,\theta,\varphi) = e^{\rho^3}$ $\quad \rho: 0 \to 1, \quad \theta: 0 \to 2\pi, \quad \varphi: 0 \to \pi$

$$\int_0^\pi \int_0^{2\pi} \int_0^1 \rho^2 e^{\rho^3} \sin\varphi \, d\rho \, d\theta \, d\varphi = \ldots = \frac{4}{3}(\pi e - \pi)$$

Ex. (31) Find the volume of the solid E that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$.



$z^2 = x^2+y^2$

$z^2 = \rho^2$

$\rho^2 \cos^2\varphi = \rho^2 \sin^2\varphi$

$1 = \dfrac{\sin^2\varphi}{\cos^2\varphi}$

$\tan\varphi = \pm 1$

$\varphi = \pm \pi/4,\ 3\pi/4$

bc $\varphi \in [0, \pi]$.

So, integral becomes

$$\int_{\pi/4}^{3\pi/4}\int_0^{2\pi}\int_0^1 \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \frac{1}{3}\int_{\pi/4}^{3\pi/4}\int_0^{2\pi} \sin\varphi \, d\theta \, d\varphi$$

$$= \frac{2\pi}{3}\int_{\pi/4}^{3\pi/4} \sin\varphi \, d\varphi$$

$$= -2\pi/3 \cdot \cos\varphi \Big|_{\pi/4}^{3\pi/4} = -\frac{2\pi}{3}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = \boxed{\frac{2\sqrt{2}\,\pi}{3}}$$