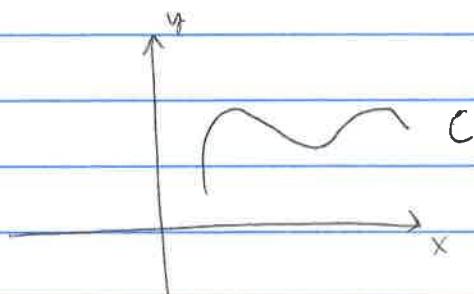


13.2 Line Integrals

Here, we want to take an integral similar to a single integral, but over a curve C (or γ) in the plane rather than just over an interval.



think of C as the curve given by the parametric equations

$$\vec{r}(t) = \langle x(t), y(t) \rangle \quad a \leq t \leq b$$

and assume that $\vec{r}$ (or C) is smooth, i.e., $\vec{r}$ is continuous and $\vec{r}'(t) \neq \vec{0}$.

If f is a smooth function defined on C , then the line integral of f along C is

$$\int_C f(x, y) ds = \lim_{\Delta s \rightarrow 0} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s$$

if the limit exists.

(and again in Ch. 10)

Recall from § 9.2 that the arc length of C is

$$L = \int_a^b ds$$

$$L = \int_a^b \|\vec{r}'(t)\| dt \quad \text{and} \quad s(t) = \int_a^t \|\vec{r}'(u)\| du$$

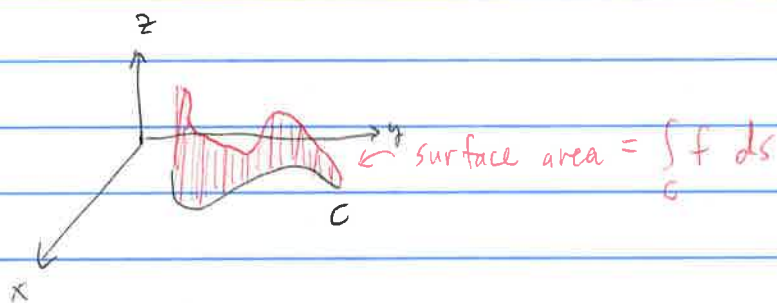
$$L = \int_a^b \sqrt{(\dot{x})^2 + (\dot{y})^2} dt$$

Q A similar argument shows that

$$\int_C f(x,y) ds = \int_a^b f(x,y) \sqrt{(\dot{x})^2 + (\dot{y})^2} dt$$

The line integral doesn't depend on the parametrization so long as the curve is only traversed once.

As with single integrals from Calc I (and beyond), we can think of line integrals of positive functions as areas over the curve C .



Ex. $\int_C (2+x^2y) ds$, C is the upper half of the unit circle
 $x^2+y^2=1$.

C is parametrized as $\vec{r}(t) = \langle \cos t, \sin t \rangle$ $t \in [0, \pi]$
 $\dot{\vec{r}}(t) = \langle -\sin t, \cos t \rangle$
 $ds = \sqrt{\sin^2 t + \cos^2 t} dt = dt$

$$\text{So } \int_C (2+x^2y) ds = \int_0^\pi 2 + \cos^2 t \sin t dt$$

$$\begin{aligned} &= \int_0^\pi 2 dt - \int_1^{-1} u^2 du \\ &= 2\pi + \frac{1}{3}(2) = \boxed{2\pi + \frac{2}{3}} \end{aligned}$$

~~Ex~~ (*) If a curve has two different pieces, then integrate over each piece separately.

Some times we will want to take the line integrals wrt x or y only, and not ds .

$$\int_C f(x,y) dx = \int_a^b f(x(t), y(t)) \dot{x}(t) dt$$

$$\int_C f(x,y) dy = \int_a^b f(x(t), y(t)) \dot{y}(t) dt$$

Frequently we take both of these at the same time (the x and y integrals occur at the same time). Then we write:

$$\int_C P(x,y) dx + \int_C Q(x,y) dy = \int_C P(x,y) dx + Q(x,y) dy$$

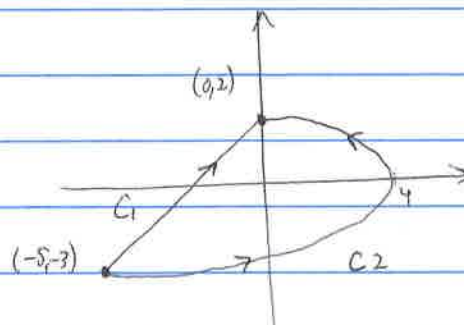
~~Ex~~

Ex. $\int_C y^2 dx + x dy$

by
where C is given piecewise:

a) C_1 = line segment from $(-5, -3)$ to $(0, 2)$

b) C_2 = segment of parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$



a) C_1 is given by $x = 5t - 5$ $0 \leq t \leq 1$
 $y = 5t - 3$

(recall: line segments are given by $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1$ $0 \leq t \leq 1$)

$$\begin{aligned} \text{so } \int_{C_1} y^2 dx + x dy &= \int_0^1 (5t-3)^2 5 dt + \int_0^1 (5t-5) 5 dt \\ &= 5 \int_0^1 25t^2 - 30t + 9 dt + 5 \int_0^1 5t - 5 dt \\ &= 5 \int_0^1 25t^2 - 25t + 4 dt \\ &= \dots = -5/6 \end{aligned}$$

b) Take y to be the parameter: $x = 4 - y^2$ $y = y$ $-3 \leq y \leq 2$

$$\begin{aligned} \int_{C_2} y^2 dx + x dy &= \int_{-3}^2 y^2 (-2y) dy + (4 - y^2) dy \\ &= \int_{-3}^2 4 - y^2 - 2y^3 dy \\ &= \left(4y - \frac{1}{3}y^3 - \frac{1}{2}y^4 \right) \Big|_{-3}^2 = \dots = \frac{245}{6} \end{aligned}$$

* Notice: These are not equal.

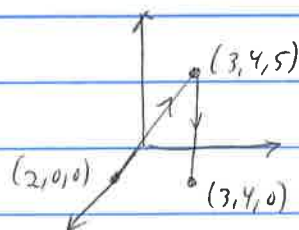
Path integrals depend on the path, in general!

Also, reversing the direction you traverse the path will give you a negative sign (just like calc I).

Path Integrals in Space: $C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ $a \leq t \leq b$

$$\begin{aligned} \int_C f(x, y, z) ds &= \int_a^b f(x, y, z) \sqrt{(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2} dt \\ &= \int_a^b f(\vec{r}) \|\dot{\vec{r}}\| dt \end{aligned}$$

Ex. $\int_C y dx + z dy + x dz$



$$C_1: \vec{r}(t) = (1-t) \langle 2, 0, 0 \rangle + t \langle 3, 4, 5 \rangle$$

$$= \langle 2+t, 4t, 5t \rangle \quad 0 \leq t \leq 1$$

$$C_2: \vec{r}(t) = \langle 3, 4, 5-5t \rangle \quad 0 \leq t \leq 1$$

Do it.

Ex. $\int_C y \sin z \, ds$ C is the helix: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$
 $0 \leq t \leq 2\pi$

Do it.

Line Integrals over vector Fields

Suppose $\vec{F}(x, y, z)$ is a continuous vector field on $\mathbb{R}^n$
 (think: force field.)

If $\vec{T}$ is the unit tangent vector field along a curve C , then the work done by ~~a particle~~ $\vec{F}$ on a particle moving along C is approximately $\vec{F} \cdot \vec{T}$ for small neighborhoods around each point. Add them all up; i.e., integrate.

So Work done by $\vec{F} = W = \int_C \vec{F}(x, y, z) \cdot \vec{T}(x, y, z) \, ds$

but $\vec{T} = \frac{\dot{\vec{r}}}{\|\dot{\vec{r}}\|}$ and $ds = \|\dot{\vec{r}}\| dt$ so this becomes

$$\int_C \vec{F}(x, y, z) \cdot \dot{\vec{r}}(x, y, z) \, dt \quad \text{or}$$

$$\boxed{\int_C \vec{F}(\vec{r}(t)) \cdot \dot{\vec{r}}(t) \, dt}$$

{ This is the line integral of $\vec{F}$ along C

$$= \int_C \vec{F} \cdot d\vec{r}$$

on a particle

Ex. Find the work done by $\vec{F}(x, y) = \langle x^2, -xy \rangle$ moving along the quarter-circle $\vec{r}(t) = \langle \cos t, \sin t \rangle$ $0 \leq t \leq \pi/2$

$$\begin{aligned} W &= \int_C \underbrace{\vec{F} \cdot \dot{\vec{r}}}_{dr} dt = \int_0^{\pi/2} \langle \cos^2 t, -\cos t \sin t \rangle \cdot \langle -\sin t, \cos t \rangle dt \\ &= \int_0^{\pi/2} -2 \cos^2 t \sin t dt \\ &= +2 \int_1^0 u^2 du = -2 \int_0^1 u^2 du = \boxed{-2/3} \end{aligned}$$

Note: $\int_{-C} \vec{F} \cdot d\vec{r} = - \int_C \vec{F} \cdot d\vec{r}$ because $\vec{T}$ is replaced by $-\vec{T}$ along $-C$.

Ex. $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle xy, yz, zx \rangle$ and C is given by $\vec{r} = \langle t, t^2, t^3 \rangle$ $0 \leq t \leq 1$

$$\text{So: } \dot{\vec{r}} = \langle 1, 2t, 3t^2 \rangle$$

$$\vec{F}(t) = \langle t^3, t^5, t^6 \rangle$$

$$\vec{F} \cdot \dot{\vec{r}} = t^3 + 2t^6 + 3t^6 = t^3 + 5t^6$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 t^3 + 5t^6 dt = \frac{1}{4} + \frac{5}{7} = \boxed{\frac{27}{28}}$$

Finally, we notice a connection between path integrals of functions and vector fields:

$$\vec{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$$

$$\vec{F}(\vec{r}(t)) = \dot{x}(t)\vec{i} + \dot{y}(t)\vec{j} + \dot{z}(t)\vec{k}, \text{ so}$$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int \left[P(x, y, z)\dot{x}(t) + Q(x, y, z)\dot{y}(t) + R(x, y, z)\dot{z}(t) \right] dt \\ &= \int_C P dx + Q dy + R dz ! \end{aligned}$$