# Calculus III: Unit I Review Guide

1. $\vec{r}_0 = Q = (0,1,2)$ , $P(1,1,1)$

$\vec{n} = \langle 3,-2,1\rangle$

$\vec{QP} = \langle 1, 0, -1\rangle$

$\vec{p} = \text{proj}_{\vec{n}}\vec{QP} = \dfrac{\vec{n}\cdot\vec{QP}}{\vec{n}\cdot\vec{n}}\langle 3,-2,1\rangle = \dfrac{3+0-1}{9+4+1}\langle 3,-2,1\rangle = \dfrac{1}{7}\langle 3,-2,1\rangle$

$$\text{dist} = \|\vec{QP} - \vec{p}\| = \|\langle 1,0,-1\rangle - \langle \tfrac{3}{7}, \tfrac{-2}{7}, \tfrac{1}{7}\rangle\| = \|\langle \tfrac{4}{7}, \tfrac{2}{7}, \tfrac{-8}{7}\rangle\|$$

$$= \frac{1}{7}\sqrt{16+4+64} = \underbrace{\frac{\sqrt{84}}{7} = \frac{\sqrt{7}\sqrt{12}}{7} = \sqrt{\frac{12}{7}} = \frac{4\sqrt{3}}{\sqrt{7}} = 4\sqrt{\frac{3}{7}}}_{\text{Any of these are fine}}$$

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2.

$\vec{n} = \vec{QP} = \langle 3, \tfrac{1}{2}, -5\rangle$

$\vec{r}_0 = P = \langle 0, \tfrac{1}{2}, 1\rangle$

<u>vector</u>: $\vec{r}(t) = \langle 3t, \tfrac{1}{2}+\tfrac{1}{2}t, 1-5t\rangle$

<u>symmetric</u>: $\dfrac{x}{3} = \dfrac{y-\frac{1}{2}}{1/2} = \dfrac{z-1}{-5}$

<u>Param</u>: $\begin{cases} x = 3t \\ y = \frac{1}{2}+\frac{1}{2}t \\ z = 1-5t \end{cases}$

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3. $\left.\begin{array}{l}\vec{n}_1 = \langle 8,-4,12\rangle \\ \vec{n}_2 = \langle 4,-2,5\rangle\end{array}\right\}$ not parallel

Intersecting? $12+8t = 1+4s$

$\Rightarrow s = \dfrac{8t+11}{4}$

$16-4t = 3-2\left(\dfrac{8t+11}{4}\right)$

$16-\cancel{4t} = 3-\cancel{4t}-\frac{11}{2}$

$16 = -5/2$ $\times$

Therefore the curves do not intersect.

Hence the lines are <u>skew</u>.

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4. $\vec{PQ} = \langle 3,-2,-4\rangle$

$\vec{PR} = \langle 1,-6,-5\rangle$

$\vec{n} = \vec{PQ}\times\vec{PR} = \langle 10-24, -4+15, -18+2\rangle$

$= \langle -14, 11, -16\rangle$

$\Pi$: $-14x+11y-16z+40=0$

$d = -\vec{n}\cdot P = -\left(-14(1)+11(2)-16(3)\right) = -\left(-14+22-48\right) = 40$

$\boxed{1}$

5. $Q(0,0,5/6)$ lies in the plane.

So does $R(1,1,0)$.

$\vec{PR} = \langle 0,4,-2\rangle$

$\vec{n} = \langle 3,2,6\rangle$

$\vec{p} = \text{proj}_{\vec{n}}\vec{PR} = \frac{\vec{n}\cdot\vec{PR}}{\vec{n}\cdot\vec{n}}\,\vec{n} = \frac{8-12}{\sqrt{9+4+36}}\,\frac{\langle 3,2,6\rangle}{\sqrt{9+4+36}}$

$\text{dist} = \|\vec{p}\| = \frac{|-4|}{\sqrt{49}} = \frac{2}{7}$

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6. $P(a,0,0)$ $\quad \vec{PQ} = \langle a,-b,0\rangle$ $\quad \vec{n} = \vec{PQ}\times\vec{PR} = \langle bc, ac, ab\rangle$

$Q(0,b,0)$ $\quad \vec{PR} = \langle a,0,-c\rangle$ $\quad d = -abc$

$R(0,0,c)$

so,

$$\boxed{\pi : (bc)x + (ac)y + (ab)z - abc = 0}$$

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7. $x^2 - 4x + 4 \;-y^2\; - (z^2 + 2z + 1) = -3 + 4 - 1$

$(x-2)^2 - y^2 - (z+1)^2 = 0$

$\Rightarrow$ $(x-2)^2 = y^2 + (z+1)^2$ Cone!

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8. WebAssign – Section 12.6

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9. $x = \frac{t^2-1}{t-1}$ $\quad$ dom$(x)$: $t \neq 1$

$y = \sqrt{t+8}$ $\quad$ dom$(y)$: $t \geq -8$

$z = \frac{\sin \pi t}{\ln t}$ $\quad$ dom$(z)$: $t > 0,\ t \neq 1$

a.) $\boxed{\text{dom}(\vec{r}) = (0,1)\cup(1,\infty)}$

$$\lim_{t\to 1} \vec{r}(t) = \lim_{t\to 1} \frac{(t+1)\cancel{(t-1)}}{\cancel{t-1}}\,\vec{i} + \lim_{t\to 1}\sqrt{t+8}\,\vec{j} + \lim_{t\to 1}\frac{\sin \pi t}{\ln t}\,\vec{k}$$

$= \frac{0}{0}$ L'Hôpital!

$$= \langle 2, 3, \lim_{t\to 1} \frac{\pi \cos \pi t}{1/t} \rangle = \langle 2, 3, -\pi\rangle$$

10. $x^2+y^2=(2)^2$ : $x=2\cos t$, $y=2\sin t$ $\qquad \vec{r}(t)=\langle 2\cos t, 2\sin t, 2\sin(2t)\rangle$

$z=xy$ : $z=4\sin t\cos t$

$\quad =2\sin(2t)$

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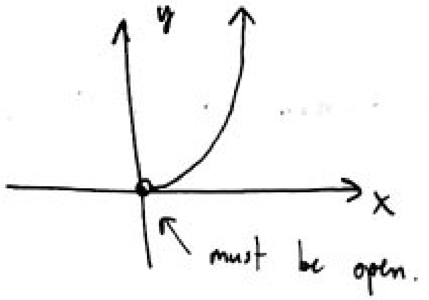
11.




$x=e^t=u$

$y=e^{2t}=(e^t)^2=u^2$ , $u>0$

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12. $\vec{u}(t)=\langle u_1(t), u_2(t), u_3(t)\rangle$

$\vec{v}(t)=\langle v_1(t), v_2(t), v_3(t)\rangle$

$(\vec{u}\cdot\vec{v})(t) = \langle u_1(t)v_1(t), u_2(t)v_2(t), u_3(t)v_3(t)\rangle$

$(\vec{u}\cdot\vec{v})^{\bullet}(t)= \langle \dot{u}_1(t)v_1(t)+u_1(t)\dot{v}_1(t), \dot{u}_2(t)v_2(t)+u_2(t)\dot{v}_2(t), \dot{u}_3(t)v_3(t)+u_3(t)\dot{v}_3(t)\rangle$

$= \langle \dot{u}_1(t)v_1(t), \dot{u}_2(t)v_2(t), \dot{u}_3(t)v_3(t)\rangle + \langle u_1(t)\dot{v}_1(t), u_2(t)\dot{v}_2(t), u_3(t)\dot{v}_3(t)\rangle$

$= \dot{\vec{u}}\cdot\vec{v} + \vec{u}\cdot\dot{\vec{v}}$ $\quad \square$

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13. $\dot{\vec{r}}(t)=\left\langle \frac{1}{1+t^2}, 4e^{2t}, 8e^t+8te^t\right\rangle$

$\dot{\vec{r}}(0)=\langle 1,4,8\rangle$ $\quad \Rightarrow \quad \boxed{\vec{T}(0)=\left\langle \frac{1}{9},\frac{4}{9},\frac{8}{9}\right\rangle}$

$\|\dot{\vec{r}}(0)\|=\sqrt{1+16+64}=\sqrt{81}=9$

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14. $\vec{r}(t)=\int\dot{\vec{r}}\,dt=\left\langle \frac{1}{2}t^2, e^t, te^t-e^t\right\rangle+\vec{c}$

$\vec{r}(0)=\langle 0,1,-1\rangle+\vec{c}=\langle 1,1,1\rangle \quad \Rightarrow \quad \vec{c}=\langle 1,0,2\rangle$

so, $\boxed{\vec{r}(t)=\left\langle \frac{1}{2}t^2+1, e^t, te^t-e^t+2\right\rangle}$

15. $\dot{x} = -e^{-t}\cos t - e^{-t}\sin t$ $\quad \dot{x}(0) = -1$

$\dot{y} = -e^{-t}\sin t + e^{-t}\cos t$ $\quad \dot{y}(0) = 1$

$\dot{z} = -e^{-t}$ $\quad \dot{z}(0) = -1$

so $\vec{v} = \langle -1, 1, -1 \rangle$

hence the tangent line is given by

$$\boxed{\frac{x-1}{-1} = \frac{y}{1} = \frac{z-1}{-1}}$$

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16. $\vec{r} \perp \dot{\vec{r}} \Rightarrow \|\vec{r}\| = \text{constant} \Rightarrow$ the terminal points of $\vec{r}$ lie on a sphere.

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17. $\dot{\vec{r}} = \langle 2, 2t, t^2 \rangle$

$$\|\dot{\vec{r}}\| = \sqrt{(t^2)^2 + 4t^2 + 4} = \sqrt{(t^2+2)^2} = t^2 + 2$$

a) $s = \int_0^t u^2 + 2 \, du = \frac{1}{3}t^3 + 2t$

b) $s(1) = \frac{1}{3} + 2 = \frac{7}{3}$.

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18.

$$\dot{\vec{r}} = \left\langle \frac{-2 \cdot 2t}{(t^2-1)^2}, \frac{(t^2+1)\cdot 2 - (2t)^2}{(t^2+1)^2} \right\rangle = \left\langle \frac{-4t}{(t^2-1)^2}, \frac{2-2t^2}{(t^2+1)^2} \right\rangle = \left\langle \frac{-4t}{(t^2-1)^2}, \frac{2(1-t^2)}{(t^2+1)^2} \right\rangle$$

$$\|\dot{\vec{r}}\| = \sqrt{\frac{16t^2}{(t^2-1)^4} + \frac{4(1-t^2)^2}{(t^2+1)^4}}$$

$$= \sqrt{\frac{16t^2(t^2+1)^4 + 4(t^2-1)^6}{(t^2-1)^4(t^2+1)^4}} = \frac{\sqrt{16t^2(t^2+1)^4 + 4(t^2-1)^6}}{(t^2-1)^2(t^2+1)^2} =$$

19. $\dot{\vec{r}} = \langle 1, 2t, 3t^2 \rangle$ $\quad \dot{\vec{r}}(1) = \langle 1, 2, 3 \rangle$ $\quad \|\dot{\vec{r}}(0)\| = \sqrt{1+4+9} = \sqrt{14}$

$\ddot{\vec{r}} = \langle 0, 2, 6t \rangle$ $\quad \ddot{\vec{r}}(2) = \langle 0, 2, 6 \rangle$ $\quad \|\dot{\vec{r}} \times \ddot{\vec{r}}\| = \| \langle 12-6, -6, 2 \rangle \|$

$= \sqrt{36+36+4} = \sqrt{76}$

So $\kappa(1) = \dfrac{\sqrt{76}}{\sqrt{14}^3} = \dfrac{2\sqrt{19}}{14\sqrt{14}} = \dfrac{\sqrt{19}}{7\sqrt{14}}$

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20. $\dot{\vec{r}} = \langle 2t, 2t^2, 1 \rangle$

$\|\dot{\vec{r}}\| = \sqrt{4t^4 + 4t^2 + 1}$

$= \sqrt{(2t^2+1)^2}$

$= 2t^2+1$

$\|\dot{\vec{r}}(1)\| = 3$

$$\boxed{\vec{T} = \left\langle \frac{2t}{2t^2+1}, \frac{2t^2}{2t^2+1}, \frac{1}{2t^2+1} \right\rangle}$$

$$\dot{\vec{T}} = \left\langle \frac{(2t^2+1)2 - 2t(4t)}{(2t^2+1)^2}, \frac{(2t^2+1)(4t) - 2t^2(4t)}{(2t^2+1)^2}, \frac{-4t}{(2t^2+1)^2} \right\rangle$$

$$= \left(\frac{1}{(2t^2+1)^2}\right) \langle 4t^2+2-8t^2, 8t^3+4t-8t^3, -4t \rangle$$

$$\dot{\vec{T}} = \left(\frac{1}{(2t^2+1)^2}\right) \langle -4t^2+2, 4t, -4t \rangle$$

$\dot{\vec{T}}(1) = \frac{1}{9}\langle -2, 4, -4 \rangle$, $\quad \|\dot{\vec{T}}(1)\| = \frac{1}{9}\sqrt{4+16+16} = \frac{\sqrt{36}}{9} = \frac{6}{9}$

$\vec{T}(1) \times \vec{N}(1) = \langle -\frac{4}{9} - \frac{2}{9}, \frac{-1}{9} + \frac{4}{9}, \frac{4}{9} + \frac{2}{9} \rangle$

$\dfrac{\|\dot{\vec{T}}(1)\|}{\|\dot{\vec{r}}(1)\|} = \dfrac{2/3}{3}$

Thus,
$$\begin{cases} \vec{T}(1) = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle \\ \vec{N}(1) = \langle \frac{-1}{3}, \frac{2}{3}, \frac{-2}{3} \rangle \\ \vec{B}(1) = \langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \rangle \\ \kappa(1) = \frac{2}{9} \end{cases}$$

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21. $\vec{r}(t) = \langle t^3, 3t, t^4 \rangle$

$\dot{\vec{r}}(t) = \langle 3t^2, 3, 4t^3 \rangle$

$\vec{n} = \langle 6, 6, -8 \rangle$

$\dot{\vec{r}} = \alpha \vec{n} \Rightarrow$ planes are parallel

$$\begin{cases} 3t^2 = 6\alpha \longrightarrow t^2 = 1 \Rightarrow t = \pm 1 \\ 3 = 6\alpha \rightarrow \alpha = \frac{1}{2} \\ 4t^3 = -8\alpha \longrightarrow t^3 = -1 \Rightarrow t = -1 \end{cases}$$

So the point is

$$\boxed{P = \vec{r}(-1) = \langle -1, -3, 1 \rangle}$$

22. $\vec{r}(t)=\langle t+2, 1-t, \frac{1}{2}t^2\rangle$

$\dot{\vec{r}}(t)=\langle 1,-1, \frac{2}{2}t\rangle$

$\|\dot{\vec{r}}\|=\sqrt{2+t^2}$

$\vec{T}=\left\langle \frac{1}{\sqrt{t^2+2}}, \frac{-1}{\sqrt{t^2+2}}, \frac{t}{\sqrt{t^2+2}} \right\rangle$

$\dot{\vec{T}}=\left\langle \frac{-1\cdot\cancel{2}t}{\cancel{2}\sqrt{t^2+2}^3}, \frac{t}{\sqrt{t^2+2}^3}, \frac{\sqrt{t^2+2}-\frac{t\cdot\cancel{2}t}{\cancel{2}\sqrt{t^2+2}}}{(\sqrt{t^2+2})^2} \right\rangle$

$=\frac{1}{\sqrt{t^2+2}^3}\langle -t, t, 2\rangle$

$\|\dot{\vec{T}}\|=\frac{1}{\sqrt{t^2+2}^3}\sqrt{t^2+t^2+4}=\frac{\sqrt{2}}{\sqrt{t^2+2}^2}$

so

$\vec{N}=\left\langle \frac{-t\sqrt{2}}{\sqrt{2}\sqrt{t^2+2}}, \frac{t\sqrt{2}}{\sqrt{2}\sqrt{t^2+2}}, \frac{2\sqrt{2}}{2\sqrt{t^2+2}} \right\rangle$

Then $\vec{T}\times\vec{N}=\frac{\sqrt{2}}{2(t^2+2)}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & t \\ -t & t & 2 \end{vmatrix}=\frac{\sqrt{2}}{2(t^2+2)}\langle -2-t^2, -t^2-2, 0\rangle=\left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right\rangle$

Therefore $\vec{B}=\left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right\rangle$ is constant. Since $\vec{B}$ is the normal vector to the osculating plane, this means that the osculating plane is constant. Hence the space curve actually lies in the single plane

$\Pi: -\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y=\frac{-3\sqrt{2}}{2}$ or $\Pi: x+y=3$.

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23. Recall that $\|\vec{T}\|=1$ so that $\|\vec{T}\|^2=\vec{T}\cdot\vec{T}=1$.

Differentiating implicitly, we obtain

$0=\frac{d}{dt}(1)=\frac{d}{dt}(\vec{T}\cdot\vec{T})=\dot{\vec{T}}\cdot\vec{T}+\vec{T}\cdot\dot{\vec{T}}=2\dot{\vec{T}}\cdot\vec{T}$

This implies $\dot{\vec{T}}\cdot\vec{T}=0$, whence $\dot{\vec{T}}\perp\vec{T}$. $\square$

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24. We did this using the definition in class. we could also do it this way:

$\vec{r}(t)=\langle a\cos t, a\sin t, 0\rangle$

$\dot{\vec{r}}(t)=\langle -a\sin t, a\cos t, 0\rangle$

$\ddot{\vec{r}}(t)=\langle -a\cos t, -a\sin t, 0\rangle$

$\dot{\vec{r}}\times\ddot{\vec{r}}=\langle 0, 0, a^2\sin^2 t+a^2\cos^2 t\rangle$
$=\langle 0, 0, a^2\rangle$

Thus,

$\kappa(t)=\frac{\|\dot{\vec{r}}\times\ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|^3}=\frac{a^2}{a^3}=\frac{1}{a}$. $\square$

25. A parabola w/ vertex at the origin is of the form $f(x) = kx^2$.

The curvature of such a parabola at the origin is given by

$$\kappa(0) = \frac{|f''(0)|}{\sqrt{1+f'(0)^2}^{\,3}} = \frac{|2k|}{\sqrt{1}^{\,3}} = |2k|$$

The radius of the osculating circle is $\frac{1}{\kappa(0)} = \frac{1}{|2k|} = \frac{1}{2}$, so

$k = \pm 1$.

Since the center of the circle is $(0, -\frac{1}{2})$, therefore the parabola must be concave down, whence $k = -1$ and

$$f(x) = -x^2.$$